Rational Normal Scrolls

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April 29, 2026

Abstract

We will give the definition of rational normal scrolls and prove some of their basic properties. In order to do this, we will go over the necessary prerequisite definitions and basic results. Then, we will discuss hyperplane sections.

1 Preliminaries

We will use the following notations:

- $[n] = \{1, \dots, n\}$.
- $[n]_0 = \{0, 1, \dots, n\}$.
- The ith entry of a column vector v is denoted v_i.
- The line passing two points P and Q is denoted $\overline{PQ}$.
- The $d-1$-dimensional linear subspace containing $P_1, \dots, P_d$ is denoted $\overline{P_1, \dots, P_d}$.

Definition 1 (Projective Space)**.** Over a field $\mathbb{K}$, the **n-dimensional projective space** $\mathbb{P}(\mathbb{K}^{n+1})$ or $\mathbb{P}^n$ is defined to be the set of equivalence classes of $(n+1)$-tuples $[x_1, \dots, x_{n+1}]$ of elements in $\mathbb{K}$, not all zero, under the equivalence relation $[x_1, \dots, x_{n+1}] \sim [\lambda x_1, \dots, \lambda x_{n+1}]$ for $\lambda \in \mathbb{K}$, $\lambda \neq 0$.

For our purposes, we will understand that $\mathbb{P}^n = \mathbb{P}(\mathbb{K}^{n+1}) \cong \mathbb{K}^n$, where $\mathbb{K}^n$ is the affine n-space over $\mathbb{K}$, and $\mathbb{P}(\mathbb{K}^{n+1} \oplus \mathbb{K}^{m+1}) \cong \mathbb{P}^{n+m+1}$, which comes from the fact that $\mathbb{K}^{n+1} \oplus \mathbb{K}^{m+1} \cong \mathbb{K}^{n+m+2}$.

Definition 2 (Homogeneous Polynomial)**.** A polynomial $P(x_1, \dots, x_n) \in \mathbb{K}[x_1, \dots, x_n]$ is **homogeneous** of degree n if for all $\lambda \in \mathbb{K}$, $P(\lambda x_1, \dots, \lambda x_n) = \lambda^n P(x_1, \dots, x_n)$.

We will only consider *bivariate* polynomials, i.e. polynomials with two variables. Denote $\mathbb{H}_n$ to be the set of degree n homogeneous bivariate polynomials. It is easy to see that $\mathbb{H}_n$ forms a $\mathbb{K}$-vector space.

Proposition 1. $\mathbb{H}_n$ has dimension $n+1$.

Proof. We can clearly see that the set $\{x^n, x^{n-1}y, \dots, xy^{n-1}, y^n\}$ forms a basis for $\mathbb{H}_n$ over $\mathbb{K}$. This set has $n+1$ polynomials, thus $\mathbb{H}_n$ has dimension $n+1$. $\square$

2 Rational Normal Curves

2.1 Definition

Definition 3 (Rational Normal Curve)**.** Let $(H_i(x, y))_{i \in [n]_0}$ be any basis of $\mathbb{H}_n$. The image of $\phi : \mathbb{P}(\mathbb{K}^2) \to \mathbb{P}(\mathbb{K}^{n+1})$ defined by $\phi(x, y) = [H_i(x, y)]_{i \in [n]_0}$ is called a **rational normal curve**.

We also say that ϕ itself is a rational normal curve, and the term *rational normal curve* will be used to denote ϕ and the image of ϕ interchangeably. We call n the *degree* of ϕ.

Example 1. Define $\phi : \mathbb{P}(\mathbb{K}^2) \to \mathbb{P}(\mathbb{K}^{n+1})$ by $\phi(x, y) = [x^n, x^{n-1}y, \cdots, y^n]$, where in this case, the basis $(H_i(x, y))_{i \in [n]_0}$ is defined by $H_i = x^{n-i}y^i$. Then the image of $\phi(x, y)$ is a rational normal curve of degree n.

Example 2. If $\phi : \mathbb{P}(\mathbb{K}^2) \to \mathbb{P}(\mathbb{K}^5)$ is defined by $\phi(x, y) = [x^4 + y^4, x^4 - y^4, x^2y^2, xy^3 + x^3y, xy^3 - x^3y]$, then the image of ϕ is a rational normal curve of degree 4.

2.2 Equivalence of Rational Normal Curves

Since any basis of $\mathbb{H}_n$ could be used to define a rational normal curve, using different bases will give us a different rational normal curve. How many of them are non-isomorphic? Equivalently, how many different rational normal curves are there, up to isomorphism? The answer is, surprisingly, there is only one up to isomorphism, i.e. all rational normal curves are isomorphic.

Lemma 1. *For a linear automorphism $A : \mathbb{K}^2 \to \mathbb{K}^2$ and a rational normal curve $\phi = \phi(x, y)$, $\phi A = \phi(A(x, y))$ is also a rational normal curve, and we can place ϕA in standard form. (That is find a basis of $\mathbb{K}_{n+1}$ so that $\phi A() = (x^{n-i}y^i)_i$).*

Proof. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

And, write polynomials as being generated by the standard dual basis to $\mathbb{K}_2$, $\{e_1, e_2\}$. And $e = (e_1, e_2)^T$.

$$\phi A(e) = [(Ae)_1^{n-i}(Ae)_2^i]_{i\in[n]_0} = [(ae_1 + be_2)^{n-i}(ce_1 + de_2)^i]_{i\in[n]_0},$$

which is homogeneous of degree n by the binomial theorem. If $B = A^{-1}$,

$$\begin{aligned} e_1^{n-i}e_2^i &= (BAe)_1^{n-i}(BAe)_2^i \\ &= (p(ae_1 + be_2) + q(ce_1 + de_2))^{n-i}(r(ae_1 + be_2) + s(ce_1 + de_2))^i \end{aligned}$$

so each $e_1^{n-i}e_2^i$ is a linear combination of $((Ae)_1^{n-i}(Ae)_2^i)_{i\in[n]_0}$, hence $((Ae)_1^{n-i}(Ae)_2^i)_{i\in[n]_0}$ is a basis. □

Let $\phi(e_1, e_2) = [e_1^2, e_1e_2, e_2^2]$ and

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{so} \quad \begin{cases} (Ae)_1 = e_1 + e_2 \\ (Ae)_2 = e_1 - e_2 \end{cases}.$$

Then

$$\begin{aligned} \phi(A) &= \phi(Ae) \\ &= [(e_1 + e_2)^2, (e_1 + e_2)(e_1 - e_2), (e_1 - e_2)^2] \\ &= [e_1^2 + e_2^2 + 2e_1e_2, e_1^2 - e_2^2, e_1^2 + e_2^2 - 2e_1e_2], \end{aligned}$$

which is also a rational normal curve.

Proposition 2. For fixed $n \in \mathbb{N}$, for any rational normal curves ϕ_1, $\phi_2 : \mathbb{P}^1 \to \mathbb{P}^n$, there is a linear automorphism $A : \mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ such that $\mathrm{im}(\phi_1) = A(\mathrm{im}(\phi_2))$.

Proof. This is simply a consequence of the polynomials in each curve being a basis for $\mathbb{H}_n$. □

We say that ϕ_1 and ϕ_2 are *projectively equivalent.* Thus all rational normal curves $\phi : \mathbb{P}^1 \to \mathbb{P}^n$ are projectively equivalent, hence for each positive integer n, there is exactly one rational normal curve ϕ, up to projective equivalence. With a change of bases on $\mathbb{K}^{n+1}$, we may write any rational normal curve as $\phi = [x^n, x^{n-1}y, \cdots, y^n]$. This is called the *standard form.*

We have defined rational normal curves, but the definition may not be too motivating, as we actually don't know why rational normal curves are special. So this section mentions some special properties of rational normal curves, without proving them.

We may consider the rational normal curves also in affine spaces. Consider the standard form $\phi(x, y) = [x^n, x^{n-1}y, \ldots, y^n]$. Since $(x, y) \in \mathbb{P}^1$, x and y are not both zero, so assume $x \neq 0$. Now letting $x = 1$ gives

$$\phi(1, 0) = [1, y, y^2, \ldots, y^n],$$

which corresponds to the point $(t, t^2, \ldots, t^n) \in \mathbb{K}^n$. As t varies, this has the shape of a moment curve.

Also, any $n + 1$ points of a rational normal curve are linearly independent in $\mathbb{P}^n$, and span $\mathbb{P}^n$. Furthermore, rational normal curve is the unique curve with this property. Intuitively, rational normal curve is the minimal degree curve to span $\mathbb{P}^n$.

2.3 Constructing $\mathbb{H}_{k+1}$ from $\mathbb{H}_k$ and $\mathbb{H}_\ell$

We wish to obtain $\mathbb{H}_{k+\ell}$ using $\mathbb{H}_k$ and $\mathbb{H}_\ell$, by getting a basis for $\mathbb{H}_{k+\ell}$ from bases of $\mathbb{H}_k$ and $\mathbb{H}_l$.

The easiest example would be using the standard bases

$$p_i = x^{k-i}y^i \quad \text{and} \quad q_j = x^{\ell-j}y^j.$$

Then

$$\{p_0q_0,\cdots,p_0q_l,p_1,q_l,\cdots,p_kq_l\} = \{x^{k+l-i}y^i\}_{i\in[k+l]_0}$$

happens to be the standard basis for $\mathbb{H}_{k+l}$. Can we generalize this beyond the standard bases?

Lemma 2. *Let $(p_i)_{i\in[k]_0}$ be a basis of $\mathbb{H}_k$ and $(q_j)_{j\in[\ell]_0}$ a basis of $\mathbb{H}_\ell$. If $V(p_0)\cap V(q_\ell)=\{0\}$, then the set*

$$\{p_0q_0,p_0q_1,\dots,p_0q_\ell,p_1q_\ell,\dots,p_kq_\ell\}$$

is a basis of $\mathbb{H}_{k+\ell}$.

p_2					p_2q_4
p_1					p_1q_4
p_0	p_0q_0	p_0q_1	p_0q_2	p_0q_3	p_0q_4
	q_0	q_1	q_2	q_3	q_4

The proof is a fun exercise, but we will be vague to keep this presentation from dragging. We induct on the number of p and q not in standard form. On the grid from the previous slide, we induct clockwise along the orange tiles, increasing the number of nonstandard polynomials. The proof is broken up into stages:

- Trivial base case where all polynomials are standard.
- Awkward transition, with only q_l arbitrary.
- Two interior cases, hand-waving permutations of the basis vectors to ensure a linear independence argument.
- Awkward transition treating all as arbitrary, now including p_0.

Proof. (Sketch) Write $r_{m,n}=\{p_1q_1,\cdots,p_1q_{l+1},p_2q_{l+1},\cdots,q_{k+1}p_{l+1}\}$ which we will induct right to left on. We setup, for $m\in[k+1]\cup\{0\}$ and $n\in[l+1]\cup\{0\}$

$$p_i = x^{k+1-i}y^{i-1} \text{ and } q_i = x^{l+1-j}y^{j-1}$$

for $i\in[m]$ and $j\in[n]$. We will induct downward on n, then downward on m. This transitions from a weak statement, to a very strong statement.

$m=k+1$ and $n=l+1$

This is the trivial case described in the example before the lemma statement.

$m=k+1$ and $n=l$

A priori, consider the value of p_1 and q_{l+1} on $(1,0)$ or $(0,1)$. They cannot both be zero, but one of them must be zero for either (why?). Then, WLOG say that our q_{l+1} features a nonzero y^l term in its standard basis representation. Then the matrix representing this new set of vectors with respect to (written as a linear sum of) the standard basis for $\mathbb{H}_{k+l}$ features nonzero entries along the entire main diagonal. The reason why this holds is trivial for the first l rows, but for the last $k+1$ this is a consequence of the nonzero y^l coefficient observation. Furthermore, the matrix is lower triangular. So it is invertible. We have a new basis.

$m=k+1$ and $n\in[l]\cup\{0\}$

This interior case has us permuting $(q_i)_{i\in[l]}$ and just forgetting that we've done that. In particular, we ensure that our ordering has q still a basis of $\mathbb{H}_l$ despite the fact we've replaced the first n of them with the standard basis vectors. The special case described above allows us to fix q_{l+1} during this selection/reordering process. At this point, we use the inductive hypothesis to write the new set of vectors, r_n, with respect to the $m+1$ spanning set (basis), r_{n+1} as a matrix. We see that this matrix is the identity matrix on all rows, except the n-th row. We focus on the top left $l+1$ by $l+1$ matrix and argue it must be invertible by our choice of ordering on q.

$m \in [k]$ and $n = 0$

This is almost identical to the previous interior case. We said "right to left," but this is more of a zig-zag. We are now focusing on the latter $k+1$ terms of $r_{m,n}$. But, we still work of right to left, removing dependency on the standard basis of H_k.

$m = 0$ and $n = 0$

We once again can argue that p_1 has a x^k or y^k term. If we assume it's x^k, then we know that p_{l+1} has a y^l term. We have the same matrix argument as before.

This concludes the induction argument. In practice, we would work in the opposite direction. We could actually use this to construct a change of basis matrix. □

3 Rational Normal Scrolls

Since we saw rational normal curves, we would like to see, roughly a rational normal *surface.*

Definition 4 (Rational Normal Scroll)**.** Let $\phi_k : \mathbb{P}^1 \to \Lambda \cong \mathbb{P}^k$ and $\phi_\ell : \mathbb{P}^1 \to \Lambda' \cong \mathbb{P}^\ell$ define two rational normal curves of degree k and ℓ, respectively. Let $A : \mathbb{P}^1 \to \mathbb{P}^1$ be a linear automorphism[1]. The **rational normal scroll** $S(k,\ell) \subset \Lambda \oplus \Lambda' \cong \mathbb{P}^{k+\ell+1}$ is the union of lines joining $\phi_k(c)$ and $\phi_\ell(Ac)$ for all $c \in \mathbb{P}^1$. That is,

$$S(k,\ell) = \bigcup_{c\in\mathbb{P}^1} \overline{\phi_k(c)\phi_\ell(Ac)}.$$

Or, with a closed formula,

$$S(k,\ell) := \{(1-t)\phi_k(c) + t\phi_l(Ac) \,|\, c \in \mathbb{P}^1, t \in \mathbb{K}\}.$$

Since we started from rational normal curves to construct scrolls, we would like to also get curves out of scrolls. Intuitively, in affine spaces, slicing out a surface gives us a curve as a cross section. A similar results hold here, saying that slicing rational normal scrolls gives cross sections that are rational normal curves.

Lemma 3. *Let $S(k,\ell)$ be a rational normal curve, and $h : \mathbb{K}^{n+1} \to \mathbb{K}$ a nonzero linear form. If $\{x \in \mathbb{K}^2 \,|\, h\phi_k(x) = h\phi_\ell(Ax) = 0\} = \{0\}$, then $\ker(h) \cap S$ is a rational normal curve.*

Here, $h : \mathbb{K}^{n+1} \to \mathbb{K}$ is a linear form, so $\ker(h)$ is a hyperplane. We call the intersection $\ker(h) \cap S$ a *hyperplane section* of S, which reflects the cross section of cutting out a surface. Our strategy is to use a coordinate system that makes h very easy to write, construct this cross section as the image of a homogeneous polynomial valued function, and then finally say that this gives a basis for $\mathbb{H}^{n-1}$. This is interpreted as saying that the cross section is a rational normal curve in the hyper-plane.

Proof. First, try to parameterize this cross section. An arbitrary element of S is written $(1-t)\phi_k + t\phi_l$ (omitting A by using our lemma). We plug this into h and set it equal to zero. Solving, we get:

$$t = \frac{h\phi_k}{h\phi_k - h\phi_l}$$

Of course this fails when the denominator is zero, but we argue that can only happen when $h\phi_k = 0 = h\phi_l$ which we assume is impossible. Parameterize $\phi : \mathbb{P}^1 \to ker(h) \cong \mathbb{P}^{n-1}$ by:

$$\phi = (h\phi_l)\phi_k + (h\phi_k)\phi_l$$

(We got rid of the denominator because it was annoying and we are in projective space.) We should at this point pick a coordinate system for Λ and Λ' so that we can write h as a very trivial row vector. Take the coordinates that have:

$$h = (1, 0, \cdots, 0, 1)$$

[1] We view A as a correspondence between the two rational normal curves.

If we write, in this new coordinate system, $\phi_k = (p_i)_i$ and $\phi_l = (q_i)_i$, then our assumption on h gives us $V(p_1) \cap V(q_{l+1}) = \{0\}$. And, in fact we write ϕ in our coordinate system:

$$\phi = (h\phi_l)\phi_k + (h\phi_k)\phi_l = (q_l p_0, \cdots, q_l p_k, p_0 q_0, \cdots, p_0 q_l)$$

And this is exactly the case of our lemma. $\square$

4 Higher-Dimensional Scrolls

4.1 Generalizing Scrolls

We have defined the rational normal scroll $S(k, \ell)$, where we used two rational normal curves $\phi_k : \mathbb{P}^1 \to \mathbb{P}^k$ and $\phi_k : \mathbb{P}^1 \to \mathbb{P}^\ell$, by taking the line passing each point on the images of ϕ_k and ϕ_ℓ. This is called a scroll of *dimension* 2

In the same manner, we would want to define a rational normal scroll using multiple rational normal curves, by taking the minimal surface passing each point on the images of each rational normal curve. We now give a general definition, involving more numbers, particularly the increasing sequence $(a_1, \ldots, a_d)$.

Definition 5 (Rational Normal Scrolls, Generalized)**.** For an increasing sequence $a_1 \leq \cdots \leq a_d$ of positive integers and $\sum a_i = n - d + 1$ we can find complementary linear subspaces $\Lambda_i \cong \mathbb{P}^{a_i} \subset \mathbb{P}^n$ and rational normal curves $C_i \subset \Lambda_i$ in each. Choose isomorphisms $\varphi_i : C_1 \to C_i$. The **rational normal scroll** of dimension d is defined as

$$S(a_1, \ldots, a_d) = \bigcup_{p \in C_1} \overline{p, \varphi_2(p), \ldots, \varphi_d(p)}.$$

Since $\mathbb{P}(\mathbb{K}^{n+1} \oplus \mathbb{K}_{m+1}) \cong \mathbb{P}(\mathbb{K}^{m+n+2}) \cong \mathbb{P}^{n+m+1}$, repeating this process will give

$$\mathbb{P}(\mathbb{K}^{a_1+1} \oplus \cdots \oplus \mathbb{K}^{a_d+1}) \cong \mathbb{P}\left(\mathbb{K}^{\sum a_i + d}\right) \cong \mathbb{P}^{\sum a_i + d - 1},$$

and it is reasonable to write this as $\mathbb{P}^n$, with letting $\sum a_i = n - d + 1$. Recall, $\overline{p, \varphi_2(p), \ldots, \varphi_d(p)}$ denotes the $d-1$-dimensional linear subspace containing p, $\varphi_2(p)$, $\ldots$, $\varphi_d(p)$. Intuitively, this is the minimal surface containing p, $\varphi_2(p)$, $\ldots$, $\varphi_d(p)$.

4.2 Generalized Slicing Lemma

Now, the original slicing lemma could be generalized to the following work, done by Brawner.

Theorem 1 (Brawner, 1997)**.** *Let $X = S(a_1, \ldots, a_d)$ be a rational normal scroll of dimension d and Y a general hyperplane section of X. Then Y is a rational normal scroll with dimension $d - 1$.*

It is easy to see when $d = 2$, then this is our original slicing lemma. If $d = 2$, since all rational normal curves are projectively equivalent, there is only one hyperplane section up to isomorphism. We are interested in the general case: how many hyperplane sections, up to isomorphism, are there? Also, what are they?

Theorem 2 (Conca-Faenzi, 2017)**.** *Given $X = S(a_1, \ldots, a_d)$, $S(b_1, \ldots, b_{d-1})$ is a hyperplane section of X if and only if the $d-1$-tuple $(b_1, \ldots, b_{d-1})$ satisfies the following:*

- $a_1 + \cdots + a_d = b_1 + \cdots + b_{d-1}$.
- $a_i \leq b_i$ *for all* $i \in [d-1]$.
- *Let* $v = \min\{j \mid a_j < b_j\}$. *Then* $b_j \geq a_{j+1}$ *for every* $j \geq v$.

4.3 Computation using Sage

So we can only look for the number of such $d-1$-tuples. This is purely combinatorial, so it would be much easier and faster to write a code in Sage to compute these hyperplane sections. Our code is the following:

```
def conca_faenzi(a):
    d = len(a)
    a_sum = sum(a)
    b_length = d - 1

    results = []

    for b in Partitions(a_sum, length=b_length): # partitions give decreasing sequences
        b = list(reversed(b)) # so we need to reverse the list to make it increasing

        # partitions already gave the first condition for the sum to be equal.

        # second condition: checking a_i =< b_i for all i
        a_less_b = all(a[i] <= b[i] for i in range(b_length))
        if not a_less_b: # if condition above is not satisfied then
            continue # discard this list and start over

        # third condition: checking a_j+1 =< b_j for all j >= v
        # we first need to find such v.
        v = -1 # defining v first, we need to find it
        for j in range(b_length): # find j such that a_j < b_j for the first time
            if a[j] < b[j]:
                v = j # replace j by v
                break

        # now check if a_j+1 =< b_j for all j >= v
        last_condition = True
        for j in range(v, b_length):
            if b[j] < a[j+1]:
                last_condition = False
                break

        if last_condition == True:
            results.append(tuple(b))
        # append: add your new result at the end

    return results

######################################
#   I N P U T    H E R E
a_tuple = (3,5,6,7)
#   I N P U T    H E R E
######################################

number = conca_faenzi(a_tuple)

print(f"Rational Normal Scroll: S{a_tuple}")
print(f"Total {len(number)} hyperplane sections:")
for res in number:
    print(f"S{res}")
```

We should mention that this code is far optimal. There should be better ways to optimize the computation time, but our current code works decently for low-dimensional (potentially < 10) scrolls.

The code logic is the following: given an increasing sequence $a_1 \leq \cdots \leq a_d$, let `a_sum` be $\sum_{i=1}^{d} a_i$. Since the increasing sequence $b_1 \leq \cdots \leq b_{d-1}$ should satisfy the first condition $\sum_{i=1}^{d} a_i = \sum_{i=1}^{d-1} b_i$, we partition `a_sum` into $d-1$ numbers. The `Partition` function in Sage automatically gives you the sequence in decreasing order, so flipping the sequence will give you an increasing sequence. After that, for each $(b_1, \ldots, b_{d-1})$, the code manually checks the second and third condition, and prints out only the sequences that satisfied all the three conditions.

Running the code with our example a 4-dimensional scroll $S(3, 5, 6, 7)$, we get total 11 hyperplane sections, printed below.

```
Rational Normal Scroll: S(3, 5, 6, 7)
Total 11 hyperplane sections:
S(3, 5, 13)
S(3, 6, 12)
S(3, 7, 11)
S(3, 8, 10)
S(5, 6, 10)
S(3, 9, 9)
S(5, 7, 9)
S(6, 6, 9)
S(5, 8, 8)
S(6, 7, 8)
S(7, 7, 7)
```

Since our input scroll was not high-dimensional, it is not hard to check that these are all the hyperplane sections, as desired.

References

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