Real Analysis 1

Taught by Minhyun Kim, Hanyang University

Joshua Im

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Contents

1 The Real Numbers

1.1 Sets and Operations on Sets

Define the following sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.
Define the following set operations: intersection, union, and complement.

Theorem 1.1

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$
- $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$

Theorem 1.2: De Morgan's Law

$(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$ and $(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$.

Definition 1.1: Cartesian Product

Define the **cartesian product** of sets A and B.

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

1.2 Functions

Theorem 1.3

- $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
- $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$

Theorem 1.4

Let $f : A \to B$. Then,

- $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
- $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$
- $f(B \setminus B_1) = A \setminus f^{-1}(B_1)$

- one-to-one: injective
- onto: subjective
- both: bijective

1.3 Mathematical Induction

Theorem 1.5: Mathematical Induction

For each $n \in \mathbb{N}$, ler $P(n)$ be a statement about the positive integer n. If

- $P(1)$ is true
- $P(k+1)$ is true whenever $P(k)$ is true

then $P(n)$ is true for all $n \in \mathbb{N}$.

Theorem 1.6: Well-Ordering Principle

Every nonempty subset of $\mathbb{N}$ has a smallest element.

Theorem 1.7: Mathematical Induction, part 2

For each $n \in \mathbb{N}$, ler $P(n)$ be a statement about the positive integer n. If

- $P(1)$ is true
- For $k > 1$, $P(k)$ is true whenever $P(j)$ is true for all positive integers $j < k$

then $P(n)$ is true for all $n \in \mathbb{N}$.

1.4 The Least Upper Bound Property

Definition 1.2: Field

A **field** $\mathbb{F}$ is a set with two operations, $+$ and $\cdot$ such that

1. If $a, b \in \mathbb{F}$, then $a + b \in \mathbb{F}$ and $a \cdot b \in \mathbb{F}$.
2. The operations are commutative: $\forall a, b \in \mathbb{F}$,
$$a + b = b + a \text{ and} a \cdot b = b \cdot a.$$
3. The operations are associative: $\forall a, b, c \in \mathbb{F}$,
$$a + (b + c) = (a + b) + c \text{ and } a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$
4. There exists an identity element 0 such that $\forall a \in \mathbb{F}$, $a + 0 = a$.
5. There exists an identity element 1 such that $\forall a \in \mathbb{F}$, $a \cdot 1 = a$.
6. $\forall a \in \mathbb{F}$, there exists an inverse element $(-a)$ such that $a + (-a) = 0$.
7. $\forall a \in \mathbb{F}$, there exists an inverse element a^{-1} such that $a \cdot a^{-1} = 1$.
8. The distributive property: $\forall a, b, c \in \mathbb{F}$,
$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

Definition 1.3: Order Properties

- O1: If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$ and $a \cdot b \in \mathbb{P}$.
- If $a \in \mathbb{R}$ then one and only one of the following hold:
$$a \in \mathbb{P}, \ -a \in \mathbb{P}, \ a = 0.$$

The positive real numbers satisfy the order properties.

Definition 1.4: Ordered Field

Any field $\mathbb{F}$ with a nonempty subset satisfying the order properties are called an **ordered field**.

Least Upper Bound of a Set

Definition 1.5: Bounded Above

A subset E of $\mathbb{R}$ is **bounded above** if there exists $\beta \in \mathbb{R}$ such that $\forall x \in E$, $x \leq \beta$. Such a β is called an **upper bound**.

We define *bounded below* and *lower bound* similarly. A set E is *bounded* if E is bounded both above and below.

Definition 1.6: Least Upper Bound

Let E be a nonempty subset of $\mathbb{R}$ that is bounded above. An element $\alpha \in \mathbb{R}$ is called the **least upper bound** or **supremum** of E if

- α is an upper bound of E
- if $\beta \in \mathbb{R}$ satisfies $\beta < \alpha$, then β is not an upper bound of E.

The second condition is equivalent to: $\alpha \leq \beta$ for all upper bounds β of E. We write $\alpha = \sup E$.

The *greatest lower bound* or *infimum* is defined similarly, and is denoted by $\inf E$.

Theorem 1.8

Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. An upper bound α of A is the supremum of A if and only if for every $\beta < \alpha$, there exists an element $x \in A$ such that

$$\beta < x \leq \alpha.$$

Proof. ($\Rightarrow$) Suppose $\alpha = \sup A$. If $\beta < \alpha$, then β is not an upper bound of A. Thus, there exists an element x in A such that $x > \beta$. On the other hand, since α is an upper bound of A, $x \leq \alpha$.

($\Leftarrow$) If α is an upper bound of A satisfying the stated condition, then every $\beta < \alpha$ is not an upper bound of A. Thus $\alpha = \sup A$. ■

Theorem 1.9: Least Upper Bound Property

Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Similarly, every nonempty subset of $\mathbb{R}$ that is bounded below has an infimum in $\mathbb{R}$.

Definition 1.7: Interval

A subset J of $\mathbb{R}$ is an **interval** if whenever $x, y \in J$ with $x < y$, then every t satisfying $x < t < y$ is in J.

We define the following specific intervals: open interval, closed interval, half-open (half-closed) intervals, and infinite intervals.

1.5 Consequences of the Least Upper Bound Property –

Theorem 1.10: Archimedean Property

If x, $y \in \mathbb{R}$ and $x > 0$, then there exists a positive integer n such that $nx > y$.

Proof. Assume $y > 0$. We prove by contradiction. Let

$$A = \{nx \mid n \in \mathbb{N}\}.$$

If the result is false, then $nx \leq y$ for $\forall n \in \mathbb{N}$, so y is an upper bound for A. Since $A \neq \emptyset$, A has a supremum in $\mathbb{R}$. Let $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$. Therefore, $\alpha - x$ is not an upper bound and this there exists an element of A, say mx such that

$$\alpha - x < mx.$$

But then $\alpha < (m+1)x$, which contradicts that α is an upper bound of A. Therefore, $\exists n$ such that $nx > y$. ∎

Corollary

Given $\epsilon > 0$, $\exists n$ such that $n\epsilon > 1$.

Theorem 1.11

If x, $y \in \mathbb{R}$ and $x < y$, then there exists $r \in \mathbb{Q}$ such that

$$x < r < y.$$

Proof. Assume $x \geq 0$. Since $y - x > 0$, $\exists n > 0$ such that

$$n(y - x) > 1 \text{ or } ny > 1 + nx.$$

also, the set $\{k \in \mathbb{N} \mid k > nx\}$ is nonempty. Then by WOP, $\exists m \in \mathbb{N}$ such that

$$m - 1 \leq nx < m.$$

Therefore,

$$nx < m \leq 1 + nx < ny,$$

and

$$x < \frac{m}{n} < y.$$

If $x < 0$ and $y > 0$, $r = 0$ works. If $x < y < 0$, then by the above there exists $r \in \mathbb{Q}$ such that $-y < r < -x$, i.e. $x < -r < y$. ∎

Theorem 1.12

Let $x > 0$ and $n \in \mathbb{N}$. Then $\exists! y \in \mathbb{R}$ such that $y^n = x$.

The proof will be postponed until a later section. For the unique y, we use the notation $y = x^{\frac{1}{n}}$.

Corollary

If a, $b > 0$ and $n \in \mathbb{N}$, then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$.

Proof is done assuming the theorem.

Proof. Let $\alpha = a^{\frac{1}{n}}$ and $\beta = b^{\frac{1}{n}}$. Then, $ab = \alpha^n \beta^n = (\alpha\beta)^n$. By uniqueness, $\alpha\beta = (ab)^{\frac{1}{n}}$. ■

Exercise 1 (1-5-6)

Prove the following:

1. Prove that between any two rational numbers, there exists an irrational number.
2. Prove that between any two real numbers, there exists an irrational number.

1.6 Countable and Uncountable Sets

Definition 1.8: Sets of the Same Cardinality

Two sets A and B are said to have the same cardinality if $\exists$ a bijective function $f : A \to B$, and denote $A \sim B$.

Definition 1.9: Finite Set

A set A is **finite** if $A \sim \{1, 2, \cdots, n\}$ for some $n \in \mathbb{N}$. A set is **infinite** if it's not finite.

Definition 1.10: Countable Set

A set A is **countable** if $A \sim \mathbb{N}$. A set is **uncountable** if it is neither finite nor countable.

Throughout this lecture, we say *at most countable* for sets that are either finite or countable.

Theorem 1.13

$\mathbb{Z}$ is countable.

Proof. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2} & 2 \mid n \\ -\frac{n-1}{2} & 2 \nmid n \end{cases}.$$

Then, since f is bijective, $\mathbb{Z}$ is countable. ■

Theorem 1.14

$\mathbb{N} \times \mathbb{N}$ is countable.

Proof. The function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m, n) = 2^{m-1}(2n-1)$ is bijective. Therefore, $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$, and $\mathbb{N} \times \mathbb{N}$ is countable. ■

Definition 1.11: Sequence

A function $f : \mathbb{N} \to A$ is called a **sequence** in A.

We use the notation $x_n = f(n)$ (the nth term of the sequence), and $\{x_n\}$.

Definition 1.12: Enumeration

Let A be a countable set and $f : \mathbb{N} - \to A$ be a bijective function. Then the sequence f is called an **enumeration** of A.

Theorem 1.15

Every infinite subset of a countable set is countable.

Proof. Let A be a countable set and $\{x_n\}$ an enumeration of A. Suppose E is an infinite subset of A. It is sufficient to construct a bijective function $f : \mathbb{N} \to E$.

Since $B = \{n \in \mathbb{N} \mid x_n \in E\}$ is nonempty, $\exists$ the minimum n_1 of B by the well-ordering principle. We define $f(1) = x_n$. Having chosen $n_1, \ldots, n_{k-1}$, we find $n_k = \min\{n > n_{k-1} \mid x_n \in E\}$ which exists by the WOP again. We define $f(k) = x_{n_k}$. Since E is infinite, f is defined on $\mathbb{N}$. We now prove that f is bijective.

First, f is one-to-one since whenever $m > k$, $n_m > n_k$. This gives if $m \neq k$, then $f(m) \neq f(k)$. Also, f is onto since for any $x = x_j \in E$, $\exists k \in \mathbb{N}$ such that $n_k = j$, and hence $f(k) = x_{n_k} = x_j = x$. ■

Theorem 1.16

If f maps $\mathbb{N}$ onto A, then A is at most countable.

Proof. If A is finite, the proof is trivial, so we assume that A is infinite. We also assume that f is not injective (one-to-one). Now, for each $a \in A$, the set $f^{-1}(a) = \{n \in \mathbb{N} \mid f(n) = a\}$ has the minimum n_a by the well ordering principle. Define a map $g : A \to \mathbb{N}$ by $g(a) = n_a$. Then, $g(A)$ is a subset of $\mathbb{N}$. Thus, g is a one-to-one function from A onto an infinite subset of $\mathbb{N}$. By the previous theorem, A is countable. ■

Definition 1.13: Indexed Families of Sets

Let A and X be nonempty sets. A function $f : A \to \mathcal{P}(X)$ is called an **indexed family** or subsets of X with an index set A.

We denote $f(\alpha)$ by E_α for each $\alpha \in A$, and f by $\{E_\alpha\}_{\alpha \in A}$.

Example 1

Let $I_n = (0, \frac{1}{n})$ for each $n \in \mathbb{N}$. Then. $\{I_n\}_{n=1}^{\infty}$ is a sequence of subsets in $\mathbb{R}$. Since $I_n \subseteq I_1$, for $\forall n \in \mathbb{N}$, we have $\displaystyle\bigcup_{n=1}^{\infty} I_n = I_1$. Show that $\displaystyle\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution Assume not. Then, $\exists x \in \displaystyle\bigcap_{n=1}^{\infty} I_n$, i.e. $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$. However, by the Archimedean property, $\exists m \in \mathbb{N}$ such that $mx > 1$ which yields a contradiction.

Exercise 2

Let $E_x = \{r \in \mathbb{Q} \mid 0 \leq r < x\}$ for each $x \in (0, 1)$. It is trivial to see that $\displaystyle\bigcup_{x \in (0,1)} E_x = E_1$. Show that $\displaystyle\bigcap_{x \in (0,1)} E_x = \{0\}$.

Theorem 1.17

Let $\{E_\alpha\}_{\alpha \in A}$ be a family of subsets of X and $E \subseteq X$. Then

- $\displaystyle\left(\bigcup_{\alpha \in A} E_\alpha\right)^{\complement} = \bigcap_{\alpha \in A} E_\alpha^{\complement}$
- $\displaystyle\left(\bigcap_{\alpha \in A} E_\alpha\right)^{\complement} = \bigcup_{\alpha \in A} E_\alpha^{\complement}$.

Theorem 1.18

Let $f : X \to Y$ be a function and let $A \neq \emptyset$.

- If $\{E_\alpha\}$ is a family of subsets of X, then
$$f\left(\bigcup E_\alpha\right) = \bigcup f(E_\alpha)$$
$$f\left(\bigcap E_\alpha\right) \subseteq \bigcap f(E_\alpha).$$
- If $\{B_\alpha\}$ is a family of subsets of Y, then
$$f^{-1}\left(\bigcup B_\alpha\right) = \bigcup f^{-1}(B_\alpha)$$
$$f^{-1}\left(\bigcap B_\alpha\right) = \bigcap f^{-1}(B_\alpha).$$

The Countability of $\mathbb{Q}$

Let $E_n = \left\{\dfrac{m}{n}; m \in \mathbb{Z}\right\}$ for each $n \in \mathbb{N}$. Then, $\mathbb{Q} = \displaystyle\bigcup_{n=1}^{\infty} E_n$. Note that each E_n is countable.

Theorem 1.19

If $\{E_n\}_{n=1}^{\infty}$ is a sequence of countable sets, then the set $S = \displaystyle\bigcup_{n=1}^{\infty} E_n$ is countable.

Proof. Let $E_n = \{x_{n,k}; k = 1, 2, \cdots\}$ be an enumeration of E_n for each $n \in \mathbb{N}$. Define a function $h : \mathbb{N} \times \mathbb{N} \to S$ by $h(n, k) = x_{n,k}$. Then h is a mapping of $\mathbb{N} \times \mathbb{N}$ onto S. Since $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$, $\exists$ a bijective function $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Thus, $h \circ g$ is a mapping of $\mathbb{N}$ onto S. Therefore, S is at most countable. Since S is infinite, S is countable. ■

Corollary : Countability of $\mathbb{Q}$

$\mathbb{Q}$ is countable.

The Uncountability of $\mathbb{R}$

Theorem 1.20: Cantor

The closed interval $[0, 1]$ is uncountable.

Proof. We use the *Cantor's diagonal argument.* Clearly, $I = [0, 1]$ is an infinite set since I contains infinitely many rational numbers. It is enough to show that every countable subset of I is a proper subset of I.

Let $E = \{x_n; n = 1, 2, \cdots\}$ be a countable subset of I.

> **Claim.** There exists a number $y \in I$ such that $y \notin E$.

Notice that each x_n has a decimal expansion

$$x_n = 0.x_{n,1}x_{n,2}x_{n,3}\cdots.$$

where $x_{n,k} \in \{0, 1, \cdots, 9\}$ for each $k \in \mathbb{N}$. Define

$$y = 0.y_1y_2y_3\cdots,$$

where

$$y_n = \begin{cases} 6 & x_{n,n} \le 5 \\ 3 & x_{n,n} \ge 6 \end{cases}.$$

Then, $y \in [0, 1]$ and it doesn't have a different decimal expansion. Moreover, since $y_n \neq x_{n,n}$ for each $n \in \mathbb{N}$, y cannot be x_n for any $n \in \mathbb{N}$. Therefore, $y \notin E$, and this completes the proof. ■

Theorem 1.21

If A is the set of all sequences whose elements are 0 or 1, then A is uncountable.

Proof. Let $E = \{s_n; n = 1, 2, \cdots\}$ be a countable subset of A. We define a new sequence $S : \mathbb{N} \times \{0, 1\}$ by

$$s(k) = 1 - s_k(k) = \begin{cases} 0 & s_k(k) = 1 \\ 1 & s_k(k) = 0 \end{cases}.$$

Clearly, $s \in A$. However, since $s(k) \neq s_k(k)$ for all $k \in \mathbb{N}$, we have $s \neq s_n$ for any $n \in \mathbb{N}$. Thus, $S \notin E$. ■

Exercise 3

If A and B are uncountable sets, then does this mean $A \sim B$? The general answer is no, and this tells that the 'uncountable' can be divided again as 'infinite' is divided to 'countable' and 'uncountable'. For example, if A is an uncountable set, then $A \nsim \mathcal{P}(A)$. Prove this result.

Note If $A \subset \mathbb{R}$ is an infinite subset, then does this mean $A \sim \mathbb{N}$ or $A \sim [0,1]$? This hypothesis is called the continuum hypothesis, and it was the first of Hilbert's 23 problems. The answer is: if cannot be proved or disproved under ZFC (Zermelo-Fraenkel set theory with the axiom of choice), proposed by Kurt Gödel and Paul Cohen.

2 Topology of the Real Line

2.1 Metric Spaces

The absolute value is defined as usual, and note the triangle inequality.

Theorem 2.1: Triangle Inequality

$|x+y| \leq |x| + |y|$.

Definition 2.1: Metric/Distance Function

Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric(distance) function** on X if

- $d(x,y) > 0$ for $\forall x,\, y \in X$
- $d(x,y) = 0 \Leftrightarrow x = y$
- $d(x,y) = d(y,x)$ for $\forall x,\, y \in X$
- $d(x,y) \leq d(x,z) + d(z,y)$ for $\forall x,\, y,\, z \in X$.

Example 1

$d(x,y) = |x-y|$ is a metric on $\mathbb{R}$. Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and define

$$d_p(x,y) = |x-y|_p = \begin{cases} \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p} & p \in [1,\infty) \\ \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\} & p = \infty \end{cases}.$$

The first three axioms are trivial, but it is not easy to prove the fourth, the triangle inequality. For $p = 2$, the triangle inequality can be proved by Cauchy-Schwarz theorem.

Example 2

Let X be the set of all bounded real-valued functions on $A(\neq \emptyset)$. We say that a function $f : A \to \mathbb{R}$ is bounded if $\exists M > 0$ such that $|f(x) \leq M$ for $\forall x \in A$. For $f,\, g \in X$, we define $d(f,g) = \sup\{|f(x) - g(x)| \mid x \in A\}$. Since

- $0 \leq |f(x) - g(x)| \leq |f(x)| + |g(x)| \leq 2M$ for $\forall x \in A$
- $d(f,g) = 0 \Leftrightarrow f = g$ since $|f(x) - g(x)| \leq d(f,g)$ for $\forall x \in A$
- $d(f,g) = d(g,f)$

The first three conditions are satisfied. For the triangular inequality, for f, g, and $h \in X$ and $x \in A$, we have

$$\begin{aligned} |f(x) - g(x)| &\leq |f(x) - h(x)| + |h(x) - g(x)| \\ &\leq d(f, h) + d(h, g), \end{aligned}$$

and hence $d(f, g) \leq d(f, h) + d(h, g)$, d is a metric on X.

2.2 Open and Closed Sets

Definition 2.2: Neighborhood

Let (X, d) be a metric space. For $\epsilon > 0$ and $p \in X$, the set

$$N_\epsilon(p) = \{x \in X \mid d(p, x) < \epsilon\}$$

is called an ϵ-**neighborhood** of p.

Definition 2.3: Interior Point

Let $E \subseteq X$. A point $p \in E$ is called an **interior point** of E if $\exists \epsilon > 0$ such that

$$N_\epsilon(p) \subseteq E.$$

We denote by $\operatorname{int}(E)$ for the set of interior points of E.

Example 3

Let $X = \mathbb{R}$ with $d(x, y) = |x - y|$, and $E = [0, 1)$. Then every p satisfying $0 < p < 1$ is an interior point of E. Indeed, if we take $\epsilon - \min\{p, 1 - p\}$, then $N_\epsilon(p) \subseteq E$. However, 0 is not an interior point since $\forall \epsilon > 0$, $N_\epsilon(0) = (-\epsilon, \epsilon)$ contains points which are not in E. Thus, $\operatorname{int}(E) = (0, 1)$.

Example 4

Let $X = [0, \infty)$ with $d(x, y) = |x - y|$ and $E = [0, 1)$. Then $(0, 1) \subseteq \operatorname{int}(E)$ as before. Moreover, $0 \in \operatorname{int}(E)$ since $N_1(0) = [0, 1) \subset E$. Therefore, $\operatorname{int}(E) = E$ in this case.

Example 5

Let $X = \mathbb{R}$ and $E = \mathbb{R} \setminus \mathbb{Q}$. Let $p \in E$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every $\epsilon > 0$ we can find $r \in \mathbb{Q} \cap N_\epsilon(p)$. Thus, no point of E is an interior point of E. Therefore, $\operatorname{int} E = \emptyset$.

Definition 2.4: Open and Closed Sets

- $O \subseteq X$ is **open** if $\text{int}(O) = O$.
- $F \subseteq X$ is **closed** if $X \setminus F$ is open.

Note $\emptyset$ and $\mathbb{R}$ are both open and closed.

Example 6

$\mathbb{Q}$ in $\mathbb{R}$ is neither open nor closed.

Theorem 2.2

Let (X, d) be a metric space.

1. If $\{O_\alpha\}_{\alpha \in A}$ is a collection of open sets of X, then $\displaystyle\bigcup_{\alpha \in A} O_\alpha$ is open.
2. If $\{O_1, \cdots, O_n\}$ is a finite collection of open sets of X, then $\displaystyle\bigcap_{j=1}^{n} O_j$ is open.

Remark.

The second statement is in general false for a countable collection of open sets. For instance, consider

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\} \text{ or } \bigcap_{n=1}^{\infty}\left(0, 1 + \frac{1}{n}\right) = (0, 1].$$

Proof. (1) We may assume that $\bigcup O_\alpha \neq \emptyset$. Let $p \in \displaystyle\bigcup_{\alpha \in A} O_\alpha$, then $p \in O_\alpha$ for some $\alpha \in A$. Since O_α is open, $\exists \epsilon > 0$ such that $N_\epsilon(p) \subseteq O_\alpha \subseteq \bigcup O_\alpha$. Thus, p is an interior point of $\displaystyle\bigcup_{\alpha \in A} O_\alpha$.

(2) Assume $\displaystyle\bigcap_{j=1}^{n} O_j \neq \emptyset$. Let $p \in \displaystyle\bigcap_{j=1}^{n} O_j$. Then $p \in O_j$ for $\forall j = 1, 2, \ldots, n$. Since each O_j is open, $\exists \epsilon_j > 0$ such that $N_{\epsilon_j}(p) \subseteq O_j$. Now, let $\epsilon = \min\{\epsilon_1, \epsilon_2, \cdots, \epsilon_j\} > 0$, then $N_\epsilon(p) \subseteq N_{\epsilon_j}(p) \subseteq O_j$ for all j. Therefore, $N_\epsilon(p) \subseteq \displaystyle\bigcap_{j=1}^{n} O_j$, and p is an interior point. ■

Definition 2.5: Limit Point and Isolated Point

Let X be a metric space, and $E \subseteq X$. A point $p \in X$ is a **limit point** of E if every ϵ-neighborhood of p contains a point $q \in E$ with $q \neq p$. A point $p \in E$ that is not a limit point of E is called an **isolated point** of E.

This says that $p \in E$ is an isolated point $\Leftrightarrow \exists \epsilon > 0$ such that $N_\epsilon(p) \cap E = \{p\}$.

Example 7

Let $E = (a, b) \subseteq \mathbb{R}$. Then every point $p \in [a, b]$ is a limit point of E.

Example 8

Let $E = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Show that each $1/n$ is an isolated point of E, and 0 is a limit point of E.

Solution If we take $\epsilon > 0$ such that

$$\epsilon < \frac{1}{n} - \frac{1}{n+1},$$

Then

$$N_\epsilon\left(\frac{1}{n}\right) \cap E = \left\{\frac{1}{n}\right\}.$$

Now, for given $\epsilon > 0$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Then, $\frac{1}{n} \in N_\epsilon(0) \cap E$, so 0 is a limit point of E. Note that 0 is the only limit point of E.

Example 9

Let $E = \mathbb{Q} \cap [0, 1] \subseteq \mathbb{R}$. Then, every $p \in [0, 1]$ is a limit point of E. Let $\epsilon > 0$. Suppose that $p \in [0, 1)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists r \in \mathbb{Q}$ such that $p < r < \min\{p + \epsilon, 1\}$. Then, $r \in N_\epsilon(p) \cap E$. Suppose now that $p = 1$. Then $\exists r \in \mathbb{Q}$ such that $\min\{0, p - \epsilon\} < r < p$. Then $r \in N_\epsilon(p) \cap E$.

Theorem 2.3

Let X be a metric space, and $F \subseteq X$. Then F is closed if and only if F contains all its limit points.

Proof. $(\Rightarrow)$ Since F^{C} is open, for every $p \in F^{\mathsf{C}}$, there is $\epsilon > 0$ such that $N_\epsilon(p) \subseteq F^{\mathsf{C}}$, i.e. $N_\epsilon(p) \cap F = \emptyset$. Thus, no point of F^{C} is a limit point of F.

$(\Leftarrow)$ To show that F^{C} is open, let $p \in F^{\mathsf{C}}$. By the assumption, p is not a limit point of F. Thus, $\exists \epsilon > 0$ such that $N_\epsilon(p) \cap F = \emptyset$. This implies $N_\epsilon(p) \subseteq F^{\mathsf{C}}$, and hence F^{C} is open. ■

Theorem 2.4

Let X be a metric space, and $E \subseteq X$. If p is a limit point of E, then every $N_\epsilon(p)$ contains infinitely many points of E.

Proof. Suppose not, the $\exists \epsilon > 0$ such that $N_\epsilon(p) \cap E$ contains only finitely many points of E. Label these points as $q_1, q_2, \ldots, q_n$ with $q_i \neq p$. Let $\epsilon_0 = \min\{d(p, q_i) \mid i = 1, 2, \cdots, n\} > 0$. Then, $N_{\epsilon_0}(p)$ contains at most p. Thus p is not a limit point of E. ∎

Corollary

A finite set in a metric space has no limit points.

Definition 2.6: Closure

Let X be a metric space, and $E \subseteq X$. Let E' be the set of limit points of E. Then, $\bar{E} = E \cup E'$ is the **closure** of E.

Theorem 2.5

1. $\bar{E}$ is closed.
2. $E = \bar{E}$ if and only if E is closed.
3. $\bar{E} \subseteq F$ for every closed set $F \subseteq X$ such that $E \subseteq F$.

Proof. (1) To prove $\bar{E}^{\complement}$ is open, let $p \in \bar{E}^{\complement}$, i.e. $p \notin E$ and $p \notin E'$. Then, $\exists \epsilon > 0$ such that $N_\epsilon(p) \cap E = \emptyset$. It is enough to show that $N_\epsilon(p) \cap E' = \emptyset$. Suppose $q \in N_\epsilon(p) \cap E'$. We choose $\delta > 0$ such that $N_\delta(q) \subseteq N_\epsilon(p)$. Since $q \in E'$, we have $N_\delta(q) \cap E \neq \emptyset$. But this implies that $N_\epsilon(p) \cap E \neq \emptyset$, which is a contradiction.

(2), (3): Exercise. ∎

Definition 2.7: Dense

Let X be a metric space, and $D \subseteq X$. Then, D is **dense** in X if $\bar{D} = X$.

Example 10

$\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Theorem 2.6

Let $U \subseteq \mathbb{R}$ be a open set. Then there exists at most countable collection $\{I_n\}$ of pairwise disjoint open intervals such that $U = \bigcup_n I_n$.

Definition 2.8: Open and Closed in a set

Let X be a metric space and $Y \subseteq X$. Then, $U \subseteq Y$ is **open in** Y if $\forall p \in U$, $\exists \epsilon > 0$ such that $N_\epsilon(p) \cap Y \subseteq U$. Also, $C \subseteq Y$ is **closed in** Y if $Y \setminus C$ is open.

Example 11

$U = [0, 1)$ is not open in $\mathbb{R}$, but open in $Y = [0, \infty)$.

2.3 Compact Sets

Definition 2.9: Compact Set

Let X be a metric space, and $E \subseteq X$. A collection $\{O_\alpha\}_{\alpha \in A}$ of open subsets of X is an open cover of E if

$$E \subseteq \bigcup_{\alpha \in A} O_\alpha.$$

A set $K \subseteq X$ is **compact** if every open cover of K has a finite subcover of K.

That is, if $\{O_\alpha\}$ is an open cover of K, then $\exists \alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subseteq \bigcup_{j=1}^{n} O_{\alpha_j}.$$

Example 12

Every finite set is compact.

Example 13

$I = (0, 1)$ is not compact. Consider $O_n = (0, \frac{n}{n+1})$ for $n \in \mathbb{N}$.. Then, $\{O_n\}_{n \in \mathbb{N}}$ is an open cover of I. Indeed, if $x \in I$, then $\exists n_0 \in \mathbb{N}$ such that $\frac{1}{n_0+1} < 1 - x$ by the Archedian property. Thus,

$$x \in O_{n_0} \subseteq \bigcup_{n=1}^{\infty} O_n.$$

But, no finite subcover can cover I. Assume to the contrary that a finite subcover $\{O_{n_1}, O_{n_2}, \ldots, O_{n_k}\}$ covers I. Let $N = \max\{n_1, \cdots, n_k\}$, then we have

$$(0,1) \subseteq \bigcup_{j=1}^{k} O_{n_j} = \left(0, \frac{N}{N+1}\right),$$

which gives a contradiction.

Exercise 4

Prove that $F = [0, \infty)$ is not compact.

Theorem 2.7

Let X be a metric space. If $K \subseteq X$ is compact, then

1. K is closed
2. If $F \subseteq K$ and F is closed, then F is compact.

Proof. (1) It is enough to show that $K^{\complement}$ is open. Let $p \in K^{\complement}$. For each $q \in K$, Let $\epsilon_q = d(p,q)/2$. Then, $N_{\epsilon_q}(p) \cap N_{\epsilon_q}(q) = \emptyset$. Since $\{N_{\epsilon_q}(q)\}_{q \in K}$ is an open cover of K, there exists $q_1, q_2, \ldots, q_n$ such that

$$K \subseteq \bigcup_{j=1}^{n} N_{\epsilon_{q_j}}(q_j).$$

Let $\epsilon = \min\{q_1, \cdots, q_n\}$. Then, $N_\epsilon(p)$ does not intersect with $N_{\epsilon_{q_j}}(q_j)$ for all $j = 1, \ldots, n$. Thus, $N_\epsilon(p) \subseteq K^{\complement}$, which proves that K is closed.

(2) Let $\{O_\alpha\}_{\alpha \in A}$ be an open cover of F. Then,

$$\{O_\alpha\}_{\alpha \in A} \cup F^{\complement}$$

is an open cover of K. Since K is compact, $\exists$ a finite subcollection of $\{O_\alpha\}_{\alpha \in A} \cup F^{\complement}$ containing K, which also contains of F. ■

Corollary

If F is closed and K is compact, then $F \cap K$ is compact.

Theorem 2.8

If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof. Suppose not. Then $\forall q \in K$, $\exists \epsilon_q > 0$ such that $N_{\epsilon_q}(q)$ contains at most one point of E. Since K is compact, we can find a subcollection of $\{N_{\epsilon_q}(q)\}_{q \in K}$ covering K, and hence E. This is a contradiction. ■

Theorem 2.9

Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of nonempty compact subsets of X such that $K_n \supseteq K_{n+1}$ for $\forall n \in \mathbb{N}$. Then, $K = \bigcap_{n=1}^{\infty} K_n$ is nonempty and compact.

Proof. Since K is a closed subset of a compact set K, it is compact. ■

2.4 Compact Subsets of $\mathbb{R}$

Theorem 2.10: Heine-Borel

Every closed and bounded interval $[a, b]$ is compact.

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of $[a, b]$. Define

$$E = \{r \in [a, b], [a, r] \text{ is covered by a finite subcover of } \mathcal{U}\}.$$

Clearly, E is nonempty and bounded. Thus $\exists \gamma = \sup E$ in $\mathbb{R}$ by the least upper bound property.

Claim. $\gamma = b$.

Suppose that $\gamma < b$. We will find a contradiction by constructing $s \in E$ such that $\gamma < s$. Since $\gamma \in U_\alpha$ for some open set $U_\alpha \in \mathcal{U}$, $\exists \epsilon > 0$ such that $N_\epsilon(\gamma) = (\gamma - \epsilon, \gamma + \epsilon) \subseteq U_\alpha$. Since $\gamma - \epsilon$ is not an upper bound of E, $\exists t \in E$ such that $\gamma - \epsilon < t < \gamma$. Thus, $[0, t]$ is covered by finitely many sets

$$U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_n}.$$

Now, choose any $s \in (\gamma, \gamma + \epsilon)$ such that $s < b$. Then,

$$[a, s] \subseteq \left(\bigcup_{j=1}^{n} U_{\alpha_j} \right) \cup U_\alpha,$$

i.e. $s \in E$. Also $\gamma \in E$ (why?), so this completes the proof. ■

Theorem 2.11: Heine-Borel-Bolzano-Weierstrass

Let $K \subseteq \mathbb{R}$. Then, the following are equal.

1. K is closed and bounded.
2. K is compact.
3. Every infinite subset of K has a limit point in K.

Proof. (1) $\Rightarrow$ (2): Since K is bounded, $\exists M > 0$ such that $K \subseteq [-M, M]$. Then, K is a closed subset of a compact set $[-M, M]$. Thus, K is compact.

(2) $\Rightarrow$ (3): Exercise.

(3) $\Rightarrow$ (1): Suppose K is not bounded, Then $\forall n \in \mathbb{N}$, $\exists p_n \in K$ such that $|p_n| > n$. We may assume that all p_i are different. Then, $\{p_n \mid n \in \mathbb{N}\}$ is an infinite subset of $\mathbb{R}$ with no limit point in $\mathbb{R}$. This is a contradiction, so K is bounded. To show that K is closed, let p be a limit point of K. Then $\forall n \in \mathbb{N}$, $\exists p_n \in K$ with $p_n \neq p$ such that $|p_n - p| < \frac{1}{n}$. Let $S = \{p_n \mid n \in \mathbb{N}\}$. Then, S is an infinite subset of K, and p is a limit point of S. Now, it is enough to show that p is the only limit point of S. ($\rightarrow p \in K$) Suppose $q \in \mathbb{R}$ with $q \neq p$. Then by the triangle inequality,

$$|q - p_n| \geq |q - p| - |p - p_n| \geq \frac{1}{2}|p - q|$$

Let $\epsilon = \frac{1}{2}|p - q|$. Then $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ by the archimedian property. Thus, $\forall n \geq N$,

$$|q - p_n| > |q - p| - \frac{1}{n} > |q - p| - \epsilon = \frac{1}{2}|p - q|.$$

Therefore, the distance between q and p_n has a positive lower bound, so q is not a limit point. ■

2.5 The Cantor Set

Let $P_0 = [0, 1]$. If we remove the middle third open interval $(\frac{1}{3}, \frac{2}{3})$, we have $P_1 = J_{1,1} \cup J_{1,2}$, where $J_{1,1} = [0, \frac{1}{3}]$ and $J_{1,2} = [\frac{2}{3}, 1]$. Now, from each of $J_{1,1}$ and $J_{1,2}$, we remove $(\frac{1}{3^2}, \frac{2}{3^2})$ and $(\frac{7}{3^2}, \frac{8}{3^2})$ so that we have $P_2 = J_{2,1} \cup J_{2,2} \cup J_{2,3} \cup J_{2,4}$. We continue this process inductively. Then,

$$P_n = \bigcup_{j=1}^{2^n} J_{n,j}.$$

where $J_{n,j}$ is a closed interval of the form $J_{n,j} = [\frac{x_j}{3^n}, \frac{x_{j+1}}{3^n}]$. Note that $P_0 \supset P_1 \supset P_2 \supset \cdots$. The set

$$P = \bigcap_{n=0}^{\infty} P_n$$

is called the Cantor set. The Cantor set satisfies some interesting properties:

1. P is nonempty and compact.

2. P contains all the endpoints of $\{J_{n,k}\}$ for $\forall n \in \mathbb{N}$ and $1 \leq k \leq 2^n$.

3. Every point of P is a limit point of P.

Proof. Let $p \in P$ and $\epsilon > 0$. Take $m \in \mathbb{N}$ sufficiently large so that $3^{-m} < \epsilon$. Since $p \in P_m$, $p \in J_{m,k}$ for some $1 \leq k \leq 2^m$. But the length of $J_{m,k}$ is 3^{-m}, so $J_{m,k} \subseteq N_\epsilon(p)$. Since both endpoints of $J_{m,k}$ are in $P \cap N_\epsilon(p)$, p is a limit point of P. ∎

4. The sum of the lengths of the intervals removed is 1.

Proof. Let the sum of the lengths be S. Then,

$$S = \frac{1}{3} + 2 \cdot \frac{1}{3^2} + 4 \cdot \frac{1}{3^3} + \cdot = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1.$$ ∎

5. P contains no intervals.

Definition 2.10: Ternary Expansion

Let $0 \leq x \leq 1$, and $n_1 = \max\{n \in \{0,1,2\} \mid \frac{n}{3} < x\}$. Having chosed $n_1, \ldots, n_k$, let

$$n_{k+1} = \max \left\{ n \in \{0,1,2\} \,\middle|\, \frac{n}{3^{k+1}} < x - \left(\frac{n_1}{3} + \frac{n_2}{3^2} + \cdots + \frac{n_k}{3^k} \right) \right\}$$

The expression $x = 0.n_1 n_2 n_3 \cdots$ is called the **ternary expression** of x.

According to the definition above, a finite expansion is not allowed. However, for the next property, we use a different convention. Assume that x has a finite expansion

$$x = \sum_{k=1}^{m} \frac{a^k}{3^k},\ a_m \in \{1,2\}.$$

If $a_n = 2$, we will use the finite expansion, and if $a_n = 1$, we will use the infinite expansion as in the previous definition, i.e.

$$x = \sum_{k=1}^{m-1} \frac{a_k}{3^k} + \sum_{k=m+1}^{\infty} \frac{2}{3^k}.$$

6. Let $x \in [0,1]$ which $x = 0.n_1 n_2 n_3 \cdots$. Then $x \in P$ if and only if $n_k \in \{0,2\}$.

7. P is uncountable.

Remark.

Note that there are countably many endpoints of $J_{n,k}$. Thus, P contains points other than the endpoints of $J_{n,k}$.

Since P has length 0, it seems like P is a 'small' set. However, the fact that P is uncountable makes the Cantor set interesting.

3 Sequences of Real Numbers

3.1 Convergent Sequences

Definition 3.1: Convergence and Limit Point

A sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space X is said to **converge** if $\forall \epsilon > 0$, $\exists p \in X$ such that $\exists n_0 \in \mathbb{N}$ such that $d(p_n, p) < \epsilon$ for $\forall n \geq n_0$. In this case, we say that $\{p_n\}$ converges to p or that p is a **limit point** of $\{p_n\}$, and write

$$\lim_{n\to\infty} p_n = p \text{ or } p_n \to p.$$

Here, $d(p_n, p) < \epsilon$ is equivalent to $p_n \in N_\epsilon(p)$ for $\forall n \geq n_0$. We say that $\{p_n\}$ **diverges** if it does not converge.

Example 1

Consider $\left\{\dfrac{2n+1}{3n+1}\right\}_{n=1}^{\infty}$. Show that $\lim_{n\to\infty} \dfrac{2n+1}{3n+1} = \dfrac{2}{3}$.

We need to find n_0 satisfying if $n > n_0$, then

$$\left|\frac{2n+1}{3n+1} - \frac{2}{3}\right| = \frac{1}{3(3n+1)} < \epsilon.$$

Solution Let $\epsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{9n_0} < \epsilon$. Then, $\forall n \geq n_0$,

$$\begin{aligned}\left|\frac{2n+1}{3n+1} - \frac{2}{3}\right| &= \frac{1}{3(3n+1)} \\ &< \frac{1}{9n} \\ &\leq \frac{1}{9n_0} \\ &< \epsilon.\end{aligned}$$

Therefore, the sequence converges to $2/3$.

Example 2

The sequence $\{1-(-1)^n\}_{n=1}^{\infty}$ diverges in $\mathbb{R}$.

Solution Suppose that $p_n = 1-(-1)^n$ converges to p for some $p \in \mathbb{R}$. Then for

$\epsilon = 1/2$, $\exists n_0 \in \mathbb{N}$ such that $|p_n - p| < \frac{1}{2}$ $\forall n \geq n_0$. However, for $n \geq n_0$,

$$\begin{aligned} 2 &= |p_n - p_{n+1}| \\ &\leq |p_n - p| + |p - p_{n+1}| \\ &< \frac{1}{2} + \frac{1}{2} \\ &= 1, \end{aligned}$$

which is a contradiction.

We also look when X is a metric space that is not the real line.

Example 3

Let $X = C[0,1]$ be the set of bounded functions on $[0,1]$ with

$$d(f,g) = \sup\{|f(x) - g(x)| \ ; \ x \in [0,1]\}.$$

Consider the sequence $\{f_n\}_{n=1}^{\infty}$ where $f_n(x) = \frac{x^n}{n}$. Then, $f_n \to 0$.

Solution Since

$$d_(f_n, 0) = \sup\left\{\left|\frac{x^n}{n} - 0\right| \ ; \ x \in [0,1]\right\} = \frac{1}{n}$$

$d(f_n, 0) \to 0$ as $n \to \infty$.

Definition 3.2: Bounded

A sequence $\{p_n\} \subseteq X$ is bounded if $\exists p \in X$ and $M > 0$ such that $d(p_n, p) \leq M$ $\forall n \in \mathbb{N}$.

Theorem 3.1

1. If $\{p_n\}$ converges in X, then the limit is unique.
2. Every convergent sequence in X is bounded.
3. If $E \subseteq X$ and p is a limit point of E, then there exists a sequence $\{p_n\}$ in E with $p_n \neq p$ such that $\lim_{n\to\infty} p_n = p$.

Proof. (1) We prove by contradiction. Suppose $\exists q_1$ and q_2 such that $q_1 \neq q_2$ and

$$q_1 = \lim_{n\to\infty} p_n = q_2.$$

Let $\epsilon = d(p_1, p_2)/2$. Since $p_n \to q_j$ $(j = 1, 2)$, $\exists n_j \in \mathbb{N}$ such that $d(p_n, q_j) < \epsilon$ $\forall n \geq$

n_j. Now, let $n_0 = \max\{n_1, n_2\}$. Then, $\forall n \geq n_0$,

$$\begin{aligned} d(q_1, q_2) &\leq d(q_1, p_n) + d(p_n, q_2) \\ &< \epsilon + \epsilon \\ &= d(q_1, q_2), \end{aligned}$$

which is a contradiction.

(2) Suppose $p_n \to p$ in X. Then $\exists n_0 \in \mathbb{N}$ such that $d(p_n, p) < 1$ $\forall n \geq n_0$. Since

$$d(p_n, p) \leq \max\{d(p_1, p), \cdots, d(p_{n_0}, p), 1\}$$

for all $n \in \mathbb{N}$, $\{p_n\}$ is bounded.

(3) For each $n \in \mathbb{N}$, $\exists p_n \in E$ with $p_n \neq p$ such that $d(p_n, p) < \frac{1}{n}$. This sequence $\{p_n\}$ converges to p. ■

Remark.

The converse of the (2) is not true in general. That is, a bounded sequence need not be convergent. For instance, the sequence $p_n = 1-(-1)^n$, is bounded but not convergent.

Remark.

In (3) of the theorem above, for example, $\sqrt{2}$ is a limit point of $\mathbb{Q}$, so $\exists\{r_n\} \subseteq \mathbb{Q}$ such that $r_n \to \sqrt{2}$. Such sequence may not be unique.

3.2 Sequences of Real Numbers

In sections 3.2 and 3.3, let $X = \mathbb{R}$ with the standard distance function.

Theorem 3.2

Suppose $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. Then,

1. $\lim\limits_{n\to\infty} (a_n + b_n) = a + b$
2. $\lim\limits_{n\to\infty} (a_n b_n) = ab$
3. If $a \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} \dfrac{b_n}{a_n} = \dfrac{b}{a}$.

Proof. (1) Let $\epsilon > 0$ be given. Then, there exists n_1, $n_2 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2} \text{ if } n \geq n_1$$

$$|b_n - b| < \frac{\epsilon}{2} \text{ if } n \geq n_2.$$

Take $n_0 = \max\{n_1, n_2\}$. Then, for $n \geq n_0$,

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \epsilon.$$

(2) Let $\epsilon > 0$ be given. Note that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &= |a_n| \cdot |b_n - b| + |a_n - a| \cdot |b|. \end{aligned}$$

Since $a_n \to a$, $\exists n_1 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2(|b| + 1)} \text{ if } n \geq n_1.$$

Also, since $\{a_n\}$ is bounded, $\exists M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Similarly, since $b_n \to b$, $\exists n_2 \in \mathbb{N}$ such that $|b_n - b| < \frac{\epsilon}{2M}$ for all $n \geq n_2$. Take $n_0 = \max\{n_1, n_2\}$, then

$$\begin{aligned} |a_n b_n - ab| &< M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2(|b| + 1)}|b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

whenever $n \geq n_0$. Therefore, $\lim_{n\to\infty} (a_n b_n) = ab$.

(3) It is enough to show that $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$. Let $\epsilon > 0$ be given. Note that

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a - a_n|}{|a| \cdot |a_n|} \text{ and } |a_n| \geq |a| - |a - a_n|.$$

Since $a \neq 0$ and $a_n \to a$, $\exists n_0 \in \mathbb{N}$ such that $|a_n - a| < |a|/2$ for all $n \geq n_0$. Then, we get $|a_n| \geq |a| - |a|/2 = |a|/2$, and hence

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2|a_n - a|}{|a|^2} \text{ if } n \geq n_0.$$

Since $a_n \to a$, $\exists n_1 \geq n_0$ such that $|a_n - a| < \epsilon \cdot \frac{|a|^2}{2}$ for all $n \geq n_1$. Therefore,

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| < \epsilon \text{ if } n \geq n_1.$$

■

Corollary

Suppose $\lim_{n\to\infty} a_n = a$. For all $c \in \mathbb{R}$, $\lim_{n\to\infty} (a_n + c) = a + c$, and $\lim_{n\to\infty} ca_n = ca$.

Theorem 3.3

Assume that $\lim\limits_{n\to 0} a_n = 0$ and that $\{b_n\}$ is bounded. Then $\lim\limits_{n\to 0} a_n b_n = 0$.

Proof. Since $\{b_n\}$ is bounded, $\exists M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Since $a_n \to 0$, $\exists n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon/M$ for all $n \geq n_0$. Then,

$$|a_n b_n - 0| < \frac{\epsilon}{M} \cdot M = \epsilon \text{ if } n \geq n_0. \qquad \blacksquare$$

Theorem 3.4: Squeeze Theorem

Suppose $a_n \leq b_n \leq c_n$ for all $n \geq n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$. Then, $\lim\limits_{n\to\infty} b_n = L$.

Proof. Let $\epsilon > 0$ be given. Then, $\exists n_1 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq n_1$, and $\exists n_2 \in \mathbb{N}$ such that $|c_n - L| < \epsilon$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then,

$$-\epsilon < a_n - L \leq b_n - L \leq c_n - L < \epsilon \text{ if } n \geq n_0,$$

so $|b_n - b| < \epsilon$, and $\lim\limits_{n\to\infty} b_n = L$. $\blacksquare$

Theorem 3.5

1. If $p > 0$, then $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$.
2. If $p > 0$, then $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$.
3. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.
4. If $p > 1$ and $\alpha \in \mathbb{R}$, then $\lim\limits_{n\to\infty} \dfrac{n^\alpha}{p^n} = 0$
5. If $|p| < 1$, then $\lim\limits_{n\to\infty} p^n = 0$
6. If $p \in \mathbb{R}$, then $\lim\limits_{n\to\infty} \dfrac{p^n}{n!} = 0$.

The last statement says that exponential functions always grow faster than polynomials.

Proof. (1) Let $\epsilon > 0$ be given. Then $\exists n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \epsilon^{1/p}.$$

Thus, for all $n \geq n_0$,

$$\left|\frac{1}{n^p} - 0\right| \leq \frac{1}{n_0^p} < \epsilon.$$

(2) Exercise.

(3) Let $x_n = \sqrt[n]{n} - 1$. Then, $\sqrt[n]{n} = 1 + x_n$ and

$$n = (1 + x_0)^n = \sum_{k=0}^{n} \binom{n}{k} x_n^k > \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2$$

if $n \geq 2$. Thus, $0 < x_n \leq \sqrt{\dfrac{2}{n-1}}$ for all $n \geq 2$. By the squeeze theorem, $x_n \to 0$ as $n \to \infty$.

(4) Since $p > 1$, we write $p = 1 + q$ for some $q > 0$. Then

$$p^n = (1+q)^n = \sum_{k=0}^{n} \binom{n}{k} q^k.$$

Choose $k_0 \in \mathbb{N}$ such that $k_0 > \alpha$. Then,

$$p^n > \binom{n}{k_0} q^{k_0} = \frac{n(n-1)\cdots(n-k_0+1)}{k_0!} q^{k_0}.$$

If $n > 2k_0$, then $n - k_0 + 1 > \frac{1}{2}n + 1 > \frac{1}{2}n$, and hence

$$\frac{n(n-1)\cdots(n-k_0+1)}{k_0!} q^{k_0} > \frac{1}{k_0!} \left(\frac{n}{2}\right)^{k_0} q^{k_0}.$$

Therefore,

$$0 \leq \frac{n^\alpha}{p^n} \leq \frac{2^{k_0} k_0!}{q^{k_0}} \cdot \frac{1}{n^{k_0 - \alpha}} \text{ for } n > 2k_0,$$

which gives $\lim\limits_{n\to\infty} \dfrac{n^\alpha}{p^n} = 0$ by the squeeze theorem.

(5), (6) Exercise. ■

Example 4

We revisit the example $\left\{\dfrac{2n+1}{3n+2}\right\}$. We have

$$\lim_{n\to\infty} \frac{2n+1}{3n+2} = \lim_{n\to\infty} \frac{2 + \frac{1}{n}}{3 + \frac{2}{n}} = \frac{2 + \lim\limits_{n\to\infty} \dfrac{1}{n}}{3 + \lim\limits_{n\to\infty} \dfrac{2}{n}} = \frac{2}{3}.$$

3.3 Monotone Sequences

Definition 3.3: Monotonically Increasing/Decreasing

A sequence $\{a_n\}$ in $\mathbb{R}$ is **monotonically increasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$, and **monotonically decreasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is **monotone** if it is either monotonically increasing or decreasing.

Definition 3.4: Strictly Increasing/Decreasing

A sequence $\{a_n\}$ in $\mathbb{R}$ is **strictly increasing** if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$, and **strictly decreasing** if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.

Theorem 3.6: Monotone Convergence Theorem

If $\{a_n\}$ is monotone and bounded, then $\{a_n\}$ converges.

Proof. Without loss of generality, assume $\{a_n\}$ is monotoically increasing. Set $E = \{a_n \mid n \in \mathbb{N}\}$. Then, $E \neq \emptyset$ and is bounded above. Let $a = \sup E$.

Claim. $\lim_{n\to\infty} a_n = a$.

Let $\epsilon > 0$ be given. Since $a - \epsilon$ is not an upper bound of E, $\exists n_0 \in \mathbb{N}$ such that $a - \epsilon < a_{n_0} \leq a$. Since $\{a_n\}$ is monotonically increasing,

$$a - \epsilon < a_n \leq a \text{ if } n \geq n_0.$$

Therefore, $|a_n - a| < \epsilon$, and $\{a_n\}$ converges. ■

Remark.

The converse of the theorem is not true. One example is $\{a_n\} = \dfrac{\sin nx}{n}$.

Corollary : Nested Interval Property

If $\{I_n\}_{n=1}^{\infty}$ is a sequence of closed and bounded intervals with $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$, then $\displaystyle\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Suppose $I_n = [a_n, b_n]$ where $a_n, b_n \in \mathbb{R}$. Since $I_n \supseteq I_{n+1}$,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n.$$

Since $\{a_n\}$ is monotonically increasing, it converges, say to a. Then since $a \leq b_n$ for all $n \in \mathbb{N}$, $a \in \displaystyle\bigcap_{n=1}^{\infty} I_n$. ■

Euler's Number

Consider a sequence $\{t_n\}_{n=1}^{\infty}$ where

$$t_n = \left(1 + \frac{1}{n}\right)^n.$$

Then the sequence $\{t_n\}$ is monotone and bounded.

First, we prove that $\{t_n\}$ is monotone. By the binomial theorem,

$$t_n = \left(1 + \frac{1}{n}\right)^n = 1 + \binom{n}{1}\frac{1}{n} + \cdots + \binom{n}{n}\frac{1}{n^n},$$

and

$$\begin{aligned} t_{n+1} &= \left(1 + \frac{1}{n+1}\right)^{n+1} \\ &= 1 + \binom{n+1}{1}\frac{1}{n+1} + \cdots + \binom{n+1}{n}\frac{1}{(n+1)^n} + \binom{n+1}{n+1}\frac{1}{(n+1)^{n+1}}. \end{aligned}$$

Since the kth term of t_{n+1} is greater than the kth term of t_n, and t_{n+1} has an extra positive term, $t_{n+1} \geq t_n$.

For boundedness, in the binomial expansion of t_n,

$$\begin{aligned} t_n &\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\ &\leq 3. \end{aligned}$$

Therefore, $\{t_n\}$ converges. The limit of $\{t_n\}$ is called the *euler's number* e, and it has a value of approximately 2.718.

Infinite Limits

Definition 3.5: Infinite Limits

Let $\{a_n\}$ be a sequence of real numbers. If $\forall M \in \mathbb{R}^+$, $\exists n_0 \in \mathbb{N}$ such that

$$a_n > M \text{ if } n \geq n_0,$$

then we say that $\{a_n\}$ diverges to ∞, and denote $a_n \to \infty$.

Theorem 3.7

If $\{a_n\}$ is monotonically increasing and not bounded above, then $a_n \to \infty$ as $n \to \infty$.

3.4 Subsequences and Bolzano-Weierstrass

Definition 3.6: Subsequence

Let (X, d) be a metric space. Given a sequence $\{p_n\}$ in X, consider a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers. Then the sequence $\{p_{n_k}\}_{k=1}^{\infty}$ is called a **subsequence** of the sequence $\{p_n\}$.

Definition 3.7: Subsequential Limit

If the subsequence $\{p_{n_k}\}$ converges, its limit is called the **subsequential limit** of the sequence $\{p_n\}$. That is, a point $p \in X$ is called a subsequential limit of the sequence $\{p_n\}$ if there exists a subsequence that converges to p.

Note that p also can be positive or negative infinity.

Remark.

Subsequential limits may not be unique.

Theorem 3.8

Let (X, d) be a metric space and let $\{p_n\}$ be a sequence in X converging to p. Then, every subsequence of $\{p_n\}$ also converges to p.

Proof. Let $\{p_{n_k}\}$ be any subsequence of $\{p_n\}$, and let $\epsilon > 0$ be given. Since $p_n \to p$, there exists a positive integer n_0 such that $d(p_n, p) < \epsilon$ if $n \geq n_0$. Since $\{n_k\}$ is strictly increasing, $n_k \geq n_0$ for all $k \geq n_0$. Therefore, $d(p_{n_k}, p) < \epsilon$ for all $k \geq n_0$. ■

Theorem 3.9

Let K be a compact subset of a metric space (X, d). Then every sequence in K has a convergent subsequence which converges in K.

Proof. Let $\{p_n\}$ be a sequence in K, and let $E = \{p_n \mid n = 1, 2, \cdots\}$. If E is finite, then there exists a point $p \in E$ and a sequence $\{n_k\}$ with $n_1 < n_2 < \cdots$ such that

$$p_{n_1} = p_{n_2} = \cdots = p.$$

The sequence $\{p_{n_k}\}$ obviously converges to $p \in K$.

If E is infinite, then E has a limit point $p \in K$. Choose n_1 such that $d(p, p_{n_1}) < 1$. Having chosen $n_1, \ldots, n_{k-1}$, choose an integer $n_k > n_{k-1}$ so that

$$d(p, p_{n_k}) < \frac{1}{k}.$$

Such an integer n_k exists since every neighborhood of p contains infinitely many points of E. The sequence $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$ converging to $p \in K$. ■

> **Remark.**
>
> The converse of the theorem is also true. That is, if K is a subset of a metric space $(X, , d)$ such that every sequence in K has a convergent subsequence, then K is compact.

Corollary : Bolzano-Weierstrass

Every bounded sequence in $\mathbb{R}$ has a convergent sequence.

Proof. Suppose $\{p_n\}$ is a bounded sequence in $\mathbb{R}$. Then there exists a positive integer M such that $\{p_n\}$ is a sequence in the compact set $[-M, M]$. The result follows from the theorem. ■

Theorem 3.10

Let $\{p_n\}$ be a sequence in a metric space (X, d). If p is a limit point of $\{p_n \mid n \in \mathbb{N}\}$, then there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $p_{n_k} \to p$ as $k \to \infty$.

3.5 Limit Superior and Inferior of a Sequence

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. For each $k \in \mathbb{N}$, define a_k and b_k by

$$a_k = \inf\{s_n \; : \; n \geq k\}$$

$$b_k = \sup\{s_n \; : \; n \geq k\}.$$

Definition 3.8: Limit Superior and Limit Inferior

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. The **limit superior** of $\{s_n\}$, denoted $\overline{\lim_{n\to\infty}} s_n$ or $\overline{\lim} s_n$, is defined as

$$\overline{\lim_{n\to\infty}} s_n = \lim_{k\to\infty} b_k = \inf_{k\in\mathbb{N}} \sup\{s_n \mid n \geq k\}.$$

Theorem 3.11

Let $\{s_n\}$ be a real sequence. Suppose $\overline{\lim} s_n \in \mathbb{R}$. Then $\beta = \overline{\lim} s_n$ if and only if $\forall \epsilon > 0$,

1. $\exists n_0 \in \mathbb{N}$ such that $s_n < \beta + \epsilon$ for all $n \geq n_0$
2. $\forall n \in \mathbb{N}$, $\exists K \geq n$ such that $s_k > \beta - \epsilon$.

Corollary

$\overline{\lim} s_n = \underline{\lim} s_n$ if and only if $\lim s_n$ exists in $[-\infty, \infty]$.

Proof. ($\Leftarrow$): Trivial

($\Rightarrow$): Suppose $\alpha = \overline{\lim} s_n = \underline{\lim} s_n \in \mathbb{R}$. Let $\epsilon > 0$. Then $\exists n_1, n_2 \in \mathbb{N}$ such that

$$s_n < \alpha + \epsilon \text{ for all } n \geq n_1 \text{ and } s_n > \alpha - \epsilon \text{ for all } n \geq n_2.$$

Thus, for any $n \geq n_0 = \max\{n_1, n_2\}$, $|s_n - \alpha| < \epsilon$, i.e. $\lim s_n = \alpha$. ∎

Exercise 5

Complete the proof for the corollary above for $\alpha = \pm\infty$.

Theorem 3.12

Let $\{a_n\}$ and $\{b_n\}$ be bounded real sequence. Then the following inequality holds.

$$\begin{aligned}\underline{\lim} a_n + \underline{\lim} b_n &\leq \underline{\lim}(a_n + b_n) \\ &\leq \underline{\lim} a_n + \overline{\lim} b_n \\ &\leq \overline{\lim}(a_n + b_n) \\ &\leq \overline{\lim} a_n + \overline{\lim} b_n.\end{aligned}$$

Proof. Exercises. ∎

Theorem 3.13

Let $\{s_n\}$ be a real sequence. Let E be the set of all subsequential limits of $\{s_n\}$ in $[-\infty, \infty]$. Then $\overline{\lim} s_n, \underline{\lim} s_n \in E$ and

1. $\overline{\lim} s_n = \sup E$
2. $\underline{\lim} s_n = \inf E$.

Proof. We first prove that $s = \overline{\lim} s_n \in E$. Assume $s \in \mathbb{R}$. It is enough to show that $\exists \{s_{n_k}\}$ such that $s_{n_k} \to s$. Let n_1 be the smallest integer such that $s - 1 < s_{n_1} < s + 1$. Having chosen $n_1 < \cdots < n_{k-1}$, let n_k be the smallest integer greater than n_{k-1} such that

$$s - \frac{1}{k} < s_{n_k} < s + \frac{1}{k}.$$

Such subsequence $\{s_{n_k}\}$ clearly converges to s.

We now prove (1). Since $s \in E$, $s \leq \sup E$. Assume that $s < \sup E$.

Case 1: $\beta = \sup E \neq \infty$. In this case, $\exists \alpha \in E$ such that $s < \alpha \leq \beta$. Take $\epsilon > 0$ sufficiently small so that $s < \alpha - 2\epsilon$. Since $s = \overline{\lim} s_n$, $\exists n_0 \in \mathbb{N}$ such that $s_n < s + \epsilon < \alpha - \epsilon$ for $\forall n \geq n_0$. But this means that there can exist only finitely many k such that $|s_k - \alpha| < \epsilon$. Thus, no subsequence of $\{s_n\}$ can converge to α, which is a contradiction.

Case 2: $\beta = \infty$: exercise. ■

Exercise 6

Complete the proof above where $s = \pm\infty$.

Example 5

If $s_n = \sin \dfrac{n\pi}{2}$, then $s_n = \begin{cases} 0 & n = 2k \\ (-1)^k & n = 2k+1 \end{cases}$

3.6 Cauchy Sequences

Definition 3.9: Cauchy Sequence

Let X be a metric space. A sequence $\{p_n\}$ is called a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $d(p_n, p_m) < \epsilon$ for all n, $m \geq n_0$.

Theorem 3.14

1. Every convergent sequence in X is Cauchy.
2. Every Cauchy sequence is bounded.

Proof. (1) Assume $p_n \to p$. Let $\epsilon > 0$. Then $\exists n_0 \in \mathbb{N}$ such that $d(p_n, p) < \epsilon/2$ for all $n \geq n_0$. Thus, $\forall n$, $m \geq n_0$,

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \epsilon.$$

(2) Assume $\{p_n\}$ is Cauchy. Then, $\exists n_0 \in \mathbb{N}$ such that $d(p_m, p_n) < 1$. For any $n \in \mathbb{N}$, we have that $d(p_n, p_{n_0}) \leq \max\{1, d(p_1, p_{n_0}), \cdots, d(p_{n-1}, p_{n_0})\}$. ■

Remark.
The converses of (1) and (2) are not true in general.

Example 6
Let $X = (0,1)$ be with $d(x,y) = |x-y|$. Consider $\{\frac{1}{n}\}_{n=1}^{\infty}$, then it is Cauchy. Indeed, $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} < \dfrac{\epsilon}{2}$. This,

$$\left|\frac{1}{n} - \frac{1}{m}\right| \le \frac{1}{n} + \frac{1}{m} < \epsilon \text{ if } n, m \ge n_0$$

However, $\dfrac{1}{n} \to 0$ in $\mathbb{R}$ and $0 \notin X$.

Example 7
Let $X = \mathbb{Q}$ be with $d(p,q) = |p-q|$. If $p_n \in X$ and $p_n \to p$ for some $p \in \mathbb{R} \setminus \mathbb{R}$. Then p_n is Cauchy, but p_n doesn't converge in X.

Lemma

Let X be a metric space. If $\{p_n\}$ is Cauchy in X that has a convergent subsequence, then $\{p_n\}$ converges.

Proof. Suppose that $p_{n_k} \to p_n$ in X. Let $\epsilon > 0$. Then $\exists k_1 \in \mathbb{N}$ such that

$$d(p_{n_k}, p) < \frac{\epsilon}{2} \text{ if } k \ge k_1.$$

Since $\{p_n\}$ is Cauchy, $\exists n_1 \in \mathbb{N}$ such that

$$d(p_n, p_m) < \frac{\epsilon}{2} \text{ if } n, m \ge n_1.$$

Let $n_0 = \max\{k_1, n_1\}$. If we take $k \ge n_0$, then $n_k \ge n_0$. Thus, $\forall n \ge n_0$, $d(p_n, p) \le d(p_n, p_{n_k}) + d(p_{n_k}, p) < \epsilon$, i.e. $p_n \to p$. ∎

Theorem 3.15

Every Cauchy sequence in $\mathbb{R}$ converges.

Proof. Let $\{p_n\}$ be Cauchy in $\mathbb{R}$. Then it is bounded. By Bolwano-Weierstrass, it has a convergent subsequence. The result follows from the lemma. ∎

Definition 3.10: Complete Metric Space

A metric space (X, d) is **complete** if every Cauchy sequence in X converges to X.

Contractive Sequences

Definition 3.11: Contractive Sequence

A sequence $\{p_n\}$ in a metric space (X, d) is **contractive** if there exists a real number $0 < b < 1$ such that

$$d(p_{n+1}, p_n) \leq bd(p_n, p_{n-1})$$

for all $n \in \mathbb{N}$, $n \geq 2$.

If $\{p_n\}$ is a contractive sequence, then

$$d(p_{n+1}, p_n) \leq b^{n-1}d(p_2, p_1),$$

and

$$d(p_{n+m}, p_n) \leq b^{n-1}d(p_2, p_1)(1 + b + \cdots + b^{m-1}) < \frac{b^{n-1}}{1-b}d(p_2, p_1).$$

Corollary

Every contractive sequence is a Cauchy sequence.

Theorem 3.16

Let (X, d) be a complete metric space. Then every contractive sequence in X converges in X. Furthermore, if the sequence $\{p_n\}$ is contractive and $p = \lim_{n\to\infty} p_n$, then

1. $d(p, p_n) \leq \dfrac{b^{n-1}}{1-b}d(p_2, p_1)$, and
2. $d(p, p_n) \leq \dfrac{b}{1-b}d(p_n, p_{n-1})$.

Proof. Exercise. ∎

3.7 Series of Real Numbers

Definition 3.12: Series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence real numbers. Define $\{s_n\}_{n=1}^{\infty}$ by

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

The sequence $\{s_n\}$ is called a **series**.

For each $n \in \mathbb{N}$, s_n is called the nth **partial sum** of the series. The series $\sum_{k=1}^{\infty} a_k$ converges/diverges if and only if the sequence $\{s_n\}$ converges/diverges.

Theorem 3.17: Cauchy Criterion

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for all $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon$$

for all $m > n > n_0$.

Proof. Since $\left| \sum_{k=n+1}^{m} a_k \right| = |s_m - s_n|$, $\{s_n\}$ is Cauchy. ■

Corollary

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Proof. Since $a_k = s_k - s_{k-1}$, the result follows from the Cauchy criterion. ■

Remark.
The converse of the corollary above is not true in general.

Theorem 3.18

Suppose $a_k \geq 0$ for all $k \in \mathbb{N}$. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\{s_n\}$ is bounded above.

Proof. Since $a_k \geq 0$ for all k, the sequence $\{s_n\}$ is monotonically increasing. Therefore, $\sum_{k=1}^{\infty} a_k = s_n$ converges if and only if $\{s_n\}$ is bounded above by monotone convergence theorem. ■

4 Limits and Continuity

4.1 Limit of a Function

Definition 4.1: Limit

Let X be a metric space, and $E \subseteq X$, and let $f : E \to \mathbb{R}$. Suppose that p is a limit point of E. The function f has a **limit** at p if $\exists L \in \mathbb{R}$ such that

$$\forall \epsilon > 0,\ \exists \delta = \delta(\epsilon, p, f) > 0 \text{ such that } |f(x) - L| < \epsilon$$

for all points $x \in E$ satisfying $0 < d(x,p) < \delta$. In this case, we write $\lim_{x\to\infty} f(x) = L$ or $f(x) \to L$ as $x \to p$.

Remark.

- If p is not a limit point, there are no points satisfying $0 < d(x,p) < \delta$ when $\delta > 0$ is sufficiently small.
- f is not necessarily defined at p.

Example 1

Consider $f(x) = \dfrac{x^2-4}{x-2}$ on $E = \mathbb{R} \setminus \{2\}$. Find $\lim_{x\to 2} f(x)$.

Solution We claim that $\lim_{x\to 2} f(x) = 4$. Let $\epsilon > 0$ and take $\delta = \epsilon$. Then for all $x \in E$ with $0 < |x-2| < \delta$,

$$|f(x) - 4| = \left|\frac{x^2-4}{x-2} - 4\right| = |x-2| < \delta = \epsilon.$$

Therefore, $\lim_{x\to 2} \dfrac{x^2-4}{x-2} = 4$.

Example 2

Let $f(x) = \dfrac{\sqrt{x+1}-1}{x}$ be defined on $E = [-1,0) \cup (0,\infty)$. Find $\lim_{x\to 0} f(x)$.

Solution We claim that $\lim_{x\to 0} f(x) = \dfrac{1}{2}$. Let $\epsilon > 0$ and take $\delta = 2\epsilon$. For $x \in E$ with $0 < |x| < \delta$,

$$\begin{aligned}\left|f(x) - \frac{1}{2}\right| &= \left|\frac{1}{\sqrt{x+1}+1} - \frac{1}{2}\right| \\ &= \left|\frac{1-\sqrt{x+1}}{2(\sqrt{x+1}+1)}\right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{-x}{2(\sqrt{x+1}+1)^2} \right| \\
&\le \frac{|x|}{2} \\
&= \frac{\delta}{2} \\
&= \epsilon.
\end{aligned}
$$

Thus, $f \to \dfrac{1}{2}$ as $x \to 0$.

> **Example 3**
>
> Let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Show that $\lim\limits_{x \to p} f(x)$ does not exist for every point $p \in \mathbb{R}$.

Solution We need to show that for every $L \in \mathbb{R}$, $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists x \in \mathbb{R}$ with $0 < |x-p| < \delta$ for which $|f(x) - L| \ge \epsilon$. We fix p, $L \in \mathbb{R}$. Take $\epsilon = \max\{|L-1|, |L|\}$, and let $\delta > 0$.

If $\epsilon = |L-1|$, take $x \in \mathbb{Q}$ such that $0 < |x-p| < \delta$. Then, $|f(x)-1| = |1-L| = \epsilon$.

If $\epsilon = |L|$, take $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $0 < |x-p| < \delta$. Then, $|f(x) - L| = |L| = \epsilon$.

Therefore, $\lim\limits_{x \to p} f(x)$ does not exist.

> **Exercise 7**
>
> Let $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ x & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Then, $\lim\limits_{x \to 0} f(x) = 0$ since $|f(x)| \le |x|$. However, $\lim\limits_{x \to p} f(x)$ does not exist for any $p \neq 0$. Show that $\lim\limits_{x \to p} f(x)$ does not exist.

> **Example 4**
>
> Let $f(x) = \dfrac{1}{x}$ be defined on $E = (0,1)$. Show that $\lim\limits_{x \to p} f(x) = \dfrac{1}{p}$ for any $p \in (0,1)$.

Solution Let $\epsilon > 0$ and take $\delta = \min\left\{\dfrac{p}{2}, \dfrac{\epsilon p^2}{2}\right\}$. Then for any point $x \in E$ with $0 < |x-p| < \delta$,

$$
\left| \frac{1}{x} - \frac{1}{p} \right| = \frac{|x-p|}{xp} < \frac{\delta}{xp} \le \frac{2\delta}{p^2} \le \epsilon.
$$

Thus, $f(x) \to \dfrac{1}{p}$ as $x \to p$.

Example 5

Let $f(x,y) = \dfrac{xy}{x^2+y^2}$ be defined on $E = \mathbb{R}^2 \setminus \{(0,0)\}$. Show that $\displaystyle\lim_{(x,y)\to(1,2)} f(x,y) = \frac{2}{5}$.

Solution Note that

$$\begin{aligned}\left|f(x,y) - \frac{2}{5}\right| &= \left|\frac{5xy - 2(x^2+y^2)}{5(x^2+y^2)}\right| \\ &= \left|\frac{(x-2y)(y-2x)}{5(x^2+y^2)}\right| \\ &\le \frac{|x-2y|}{5(x^2+y^2)}|y-2+2(1-x)| \\ &\le \frac{|x|+2|y|}{5(x^2+y^2)}\left(2|x-1|+|y-2|\right).\end{aligned}$$

Let $\epsilon > 0$, and let $\delta = \left(0, \dfrac{1}{2}\right)$ be a constant to be determined. If $(x,y) \in E$ satisfies $0 < d\left((x,y),(1,2)\right) < \delta$, then $\dfrac{1}{2} < |x| < \dfrac{3}{2}$ and $\dfrac{3}{2} < |y| < \dfrac{5}{2}$. Thus,

$$\frac{|x|+2|y|}{5(x^2+y^2)}\left(2|x-1|+|y-2|\right) \le \frac{\frac{3}{2}+2\cdot\frac{5}{2}}{5(\frac{1}{4}+\frac{9}{4})}(2\delta+\delta) = \frac{39}{25}\delta.$$

By taking $\delta = \min\left\{\dfrac{1}{2}, \dfrac{25}{39}\epsilon\right\}$, we obtain that

$$\frac{|x|+2|y|}{5(x^2+y^2)}\left(2|x-1|+|y-2|\right) < \epsilon.$$

Theorem 4.1

Let X be a metric space, $E \subseteq X$, p a limit point of E, and $f : E \to \mathbb{R}$. Then $\displaystyle\lim_{x\to p} f(x) = L$ if and only if $\displaystyle\lim_{n\to\infty} f(p_n) = L$ for every sequence $\{p_n\}$ in E with $p_n \ne p$ such that $\displaystyle\lim_{n\to\infty} p_n = p$.

Proof. ($\Rightarrow$) Let $\{p_n\}$ be any sequence in E such that $p_n \neq p$ and $\displaystyle\lim_{n\to\infty} p_n = p$. Let $\epsilon > 0$. Since $\displaystyle\lim_{x\to p} f(x) = L$, $\exists \delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in E$ with $0 < |x-p| < \delta$. Since $\displaystyle\lim_{n\to\infty} p_n = p$, $\exists n_0 \in \mathbb{N}$ such that $0 < |p_n - p| < \delta$ for $\forall n \ge n_0$. This, $|f(p_n) - L| < \epsilon$ for all $n \ge n_0$.

($\Leftarrow$) Assume to the contrary that $\displaystyle\lim_{x\to p} f(x) \ne L$. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$,

$\exists x \in E$ with $0 < |x-p| < \delta$ and $|f(x)-L| \geq \epsilon$. In particular, for $\delta = 1/n$ ($n \in \mathbb{N}$), $\exists p_n \in E$ such that $0 < |p_n - p| < 1/n$ and $|f(p_n) - L| \geq \epsilon$. This means that $\lim_{n\to\infty} f(p_n) \neq L$, which is a contradiction. ■

Corollary

If f has a limit at p, then it is unique.

We now look for the application of the theorem.

Example 6

Let $f(x) = \sin \frac{1}{x}$ be defined on $E = (0, \infty)$. Let $p_n = \frac{2}{(2n+1)\pi}$. Then $f(p_n) = (-1)^n$, which is oscillating. Since $\lim_{n\to\infty} f(p_n)$ does not exist, $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Theorem 4.2

Suppose E is a subset of a metric space X, f, $g : R \to \mathbb{R}$, and p is a limit point of E. If

$$\lim_{x\to p} f(x) = A \text{ and } \lim_{x\to p} g(x) = B,$$

then

1. $\lim_{x\to p} \big(f(x) + g(x)\big) = A + B$
2. $\lim_{x\to p} f(x)g(x) = AB$
3. $\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{A}{B}$ if $B \neq 0$.

Proof. (1), (2) Applying theorem 4.1 reduces the proof of sequences, which were already done at chapter 3.

(3) By (2) it suffices to show that

$$\lim_{x\to p} \frac{1}{g(x)} = \frac{1}{B}.$$

Claim. $g(x) \neq 0$ for all x sufficiently close to p, $x \neq p$.

Take $\epsilon = |B|/2$. Then there exists $\delta_1 > 0$ such that $|g(x) - B| < |B|/2$ for all $x \in E < 0 < |x-p| < \delta_1$. We now have $|g(x) - B| \geq ||g(x)| - |B||$, so

$$|g(x)| > |B| - \frac{|B|}{2} = \frac{|B|}{2} > 0$$

for all $x \in E$, $0 < |x - p| < \delta_1$.

Apply theorem 4.1 and the limit of the reciprocal for sequences. Let $\{p_n\}$ be any sequence in E with $p_n \to p$ and $p_n \neq p$ for all n. Then there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $0 < |p_n - p| < \delta_1$. Thus $g(p_n) \neq 0$ for all $n \geq n_0$. Therefore

$$\lim_{n\to\infty} \frac{1}{g(p_n)} = \frac{1}{B},$$

and this holds for every sequence $\{p_n\}$ such that $p_n \to p$, so

$$\lim_{x\to p} \frac{1}{g(x)} = \frac{1}{B}.$$

■

Definition 4.2: Bounded Function

A real-valued function f defined on a set E is **bounded** on E is there exists a constant M such that $|f(x)| \leq M$ for all $x \in E$.

Proof. Exercise. ■

Theorem 4.3

If g is bounded on E and $\lim_{x\to p} f(x) = 0$, then

$$\lim_{x\to p} f(x)g(x) = 0.$$

Proof. Exercise. ■

Theorem 4.4: Squeeze Theorem

Suppose f, g, and h are functions from E to $\mathbb{R}$ satisfying

$$g(x) \leq f(x) \leq h(x) \text{ for all } x \in E.$$

If $\lim_{x\to p} g(x) = \lim_{x\to p} h(x) = L$, then $\lim_{x\to p} f(x) = L$.

Definition 4.3: Limits at Infinity

The function $f : E \to \mathbb{R}$ has a **limit at infinity** if there exists a number $L \in \mathbb{R}$ such that given $\epsilon > 0$, $\exists M$ such that

$$|f(x) - L| < \epsilon \text{ for all } x \in E \cap (M, \infty).$$

4.2 Continuous Functions

Definition 4.4: Continuous

Let X be a metric space, and $E \subseteq X$. A function $f : E \to \mathbb{R}$ is continuous at a point $p \in E$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(p)| < \epsilon$ for all $x \in E$ with $d(x, p) < \delta$.

Theorem 4.5

If f, $g : E \subseteq X \to \mathbb{R}$ are continuous at $p \in E$, then $f \pm g$ and fg are also continuous at p. If $g(x) \neq 0$ for all $x \in E$, f/g is also continuous at p.

Proof. If p is an isolated point of EE, then the result is true since every function on E is continuous at p. If p is a limit point of E, then the conclusions follow from theorem 4.2. ■

Theorem 4.6

Let A, $B \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ be functions such that the range of f is in B. If f is continuous at $p \in A$ and g is continuous at $f(p)$, then $h = g \circ f$ is continuous at p.

Proof. Fix $\epsilon > 0$. Since g is continuous at $f(p)$, $\exists \delta_1 > 0$ such that

$$|g(y) - g(f(p))| < \epsilon \quad \text{for all} \quad y \in B \cap N_{\delta_1}(f(p)).$$

Since f is continuous at p, $\exists \delta > 0$ such that

$$|f(x) - f(p)| < \delta_1 \quad \text{for all} \quad x \in A \cap N_\delta(p).$$

Thus if $x \in A$ with $|x - p| < \delta$,

$$|h(x) - h(p)| = |g(f(x)) - g(f(p))| < \epsilon,$$

so h is continuous at p. ■

Example 7

If p is a polynomial function of degree n, defined by

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and a_0, a_1, $\ldots$, $a_n \in \mathbb{R}$ with $a_n \neq 0$, then $p(x)$ is continuous on $\mathbb{R}$.

Example 8

Suppose p and q are two polynomials on $\mathbb{R}$ and $E = \{x \in \mathbb{R} : q(x) = 0\}$. Then the rational function r defined on $\mathbb{R} \setminus E$ by

$$r(x) = \frac{p(x)}{q(x)}, \qquad x \in \mathbb{R} \setminus E,$$

is continuous on $\mathbb{R} \setminus E$.

Theorem 4.7: Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a) < \gamma < f(b)$ for some $\gamma \in \mathbb{R}$, then $\exists c \in (a, b)$ such that $f(c) = \gamma$.

Proof. Let $A = \{x \in [a, b] : f(x) \leq \gamma\}$. We have $A \neq \emptyset$ since $a \in A$, and A is bounded above by b. Thus A has a supremum in $\mathbb{R}$, denote it by c. Then $c \leq b$.

Claim. $f(c) = \gamma$.

Suppose $f(c) < \gamma$. Let $\epsilon = (\gamma - f(c))/2 > 0$. Since f is continuous at c, $\exists \delta > 0$ such that

$$f(c) - \epsilon < f(x) < f(c) + \epsilon \quad \text{for all} \quad x \in N_\delta(c) \cap [a, b].$$

Since $f(c) < \gamma$, $c \neq b$, and thus $(c, b] \cap N_\delta(c) \neq \emptyset$. But for any $x \in (c, b]$ with $c < x < c + \delta$,

$$f(x) < f(c) + \epsilon = f(c) + \frac{1}{2}\gamma - \frac{1}{2}f(c) = \frac{1}{2}(f(c) + \gamma) < \gamma.$$

But then $x \in A$ and $x > c$, contradicting $c = \sup A$. Therefore $f(c) \geq \gamma$.

Since $c = \sup A$, either $c \in A$ or c is a limit point of A. If $c \in A$, then $f(c) \leq \gamma$. If c is a limit point of A, then there exists a sequence $\{x_n\}$ in A such that $x_n \to c$. Since $x_n \in A$, $f(x_n) \leq \gamma$. Since f is continuous,

$$f(c) = \lim_{n\to\infty} f(x_n) \leq \gamma.$$

Therefore $f(c) = \gamma$. ∎

Corollary

f $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is continuous on I, then $f(I)$ is an interval.

Proof. Let $s, t \in f(I)$ with $s < t$, and let $a, b \in I$ with $a \neq b$ be such that $f(a) = s$ and $f(b) = t$. Suppose γ satisfies $s < \gamma < t$. If $a < b$, then since f is continuous on

$[a, b]$, by the intermediate value theorem there exists $c \in (a, b)$ such that $f(c) = \gamma$. Thus $\gamma \in f(I)$. A similar argument also holds if $a > b$. ∎

4.3 Uniform Continuity

Definition 4.5: Uniformly Continuous

Let $E \subseteq X$ and $f : E \to \mathbb{R}$. The function f is **uniformly continuous** on E if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in E$ with $d(x, y) < \delta$.

Example 9

If $f(x) = \sin x$, then since $|f(x) - f(y)| \leq |x - y|$ for $\forall x, y \in \mathbb{R}$, $\sin x$ is uniformly continuous on $\mathbb{R}$.

Example 10

Show that $f(x) = \dfrac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Solution Suppose that it is uniformly continuous. Then for $\epsilon = 1$, $\exists \delta \in (0, 1)$ such that $\left|\dfrac{1}{x} = \dfrac{1}{y}\right| < 1$ $\forall x, y \in (0, 1)$ wuth $|x - y| < \delta$.

Choose $x \in \left(0, \dfrac{1}{2}\right)$ and $y = x + \dfrac{1}{2}\delta \in (0, 1)$. Since $|x - y| = \dfrac{1}{2}\delta < \delta$, we have

$$1 > \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x - y|}{xy} > \frac{\delta/2}{x}$$

and hence $\dfrac{1}{2}\delta > x$. Since $x \in \left(0, \dfrac{1}{2}\right)$ was arbitrary, we arrive at a contradiction.

Example 11

Show that $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[0, \infty)$ for all $a > 0$.

Definition 4.6: Lipschitz Functions

A function $f : E \subseteq X \to \mathbb{R}$ is **Lipschitz** is $\exists M > 0$ such that $|f(x) - f(y)| \leq Md(x, y)$.

Theorem 4.8

Any Lipschitz function is uniformly continuous.

Proof. Exercise. ∎

Exercise 8

Show that $f(x) = \sqrt{x}$ on $(0, \infty)$ is uniformly continuous but not Lipschitz continuous.

Theorem 4.9: Uniform Continuity Theorem

If $K \subseteq X$ is compact and $f : K \to \mathbb{R}$ is continuous on K, then f is uniformly continuous on K.

Proof. Let $\epsilon > 0$. Since f is continuous at each point of $p \in K$, $\exists \delta_p > 0$ such that $|f(x) - f(p)| < \epsilon/2$ for all $x \in K \cap N_{\delta_p}(p)$. Then $\{N_{\delta_p/2}(p)\}_{p \in K}$ forms an open cover of K. Since K is compact, $\exists p_1, \ldots, p_k$ such that

$$K \subseteq \bigcup_{j=1}^{n} N_{\delta_{p_j}/2}(p_j).$$

Let $\delta = \min\{\delta_{p_j}/2 : j = 1, \cdots, n\} > 0$. If x, $y \in K$ and $d(x, y) < \delta$, then $x \in N_{\delta_{p_j}/2}(p_j)$ for some j. Moreover,

$$d(p_j, y) \le d(p_j, x) + d(x, y) \le \delta_{p_j}/2 + \delta < \delta_{p_j}.$$

Thus, x, $y \in N_{\delta_{p_j}}(p_j)$. By the triangle inequality,

$$|f(x) - f(y)| \le |f(x) - f(p_j)| + |f(p_j) - f(y)| < \epsilon.$$ ∎

Corollary

A continuous function on $[a, b]$ is uniformly continuous.

This is true by Heine-Borel. Here, both boundedness and closedness of $[a, b]$ are required.

Exercise 9

Show that $f(x) = x^2$ on $[0, \infty)$ is not uniformly continuous.

4.4 Monotone Functions and Discontinuities

Definition 4.7: Left and Right Limits

Let $E \subseteq \mathbb{R}$ and let p be a limit point of $E \cap (p, \infty)$. A function $f : E \to \mathbb{R}$ has a **right limit** at p if $\exists L \in \mathbb{R}$ with $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in E \cap (p, p + \delta)$. In this case, we write

$$f(p+) = \lim_{x \to p^+} f(x) = \lim_{\substack{x \to p \\ x > p}} f(x).$$

Left limits are defined similarly.

Let p be a limit point of E. Then f has a limit at p if and only if

- $f(p+)$ and $f(p-)$ both exist
- $f(p+) = f(p-)$.

Definition 4.8: Right and Left Continuities

Let $E \subseteq \mathbb{R}$. A function $f : E \to \mathbb{R}$ is **right continuous** at $p \in E$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(p)| < \epsilon$ for all $x \in E \cap [p, p + \delta)$.

Left continuity is defined similarly.

A function $f : (a, b) \to \mathbb{R}$ is right continuous at $p \in (a, b)$ if and only if $f(p+)$ exists and equals to $f(p)$.

Note A function $f : (a, b)$ is continuous at $p \in (a, b)$ if and only if

- $f(p+)$ and $f(p-)$ both exist
- $f(p+) = f(p-) = f(p)$.

So for a function not to be continuous, either $f(p+)$, $f(p-)$, or $f(p)$ does not exist, or if they all exist but not equal. There is a name for these discontinuities.

Definition 4.9: Jump Discontinuity

A function $f : I \to \mathbb{R}$ has a **jump discontinuity** at $p \in \text{int}(I)$ if $f(p+)$ and $f(p-)$ both exist, but f is not continuous at p.

Jump discontinuities are also referred to as **discontinuities of the first kind**. All other discontinuities are said to be of **second kind**.

If $f(p+) \neq f(p-)$, f has a jump discontinuity at p. If $f(p+) = f(p-) \neq f(p)$, the discontinuity is removable. All discontinuities for which $f(p+)$ or $f(p-)$ does not exist are discontinuities of the second kind.

Example 12

Let $g(x) = \begin{cases} \dfrac{x^2-4}{x=2} & x \neq 2 \\ 2 & x = 2 \end{cases}$. Then this function has a removable discontinuity at $x = 2$. Here, we can redefine $g(2) = 4$ so that g is continuous at $x = 2$.

Example 13

Let $h(x) = x \sin \dfrac{1}{x}$ on $(0, \infty)$. Then this function has a removable discontinuity at $x = 0$. Here, we can define $h(0) = 0$ so that h is continuous at 0.

Example 14

Let $f(x) = [x]$, the greatest integer function. Then f has a jump discontinuity at each $n \in \mathbb{Z}$ and is continuous on $\mathbb{R} \setminus \mathbb{Z}$.

Example 15

Consider $f(x) = \begin{cases} 0 & x \leq 0 \\ \sin \dfrac{1}{x} & x > 0 \end{cases}$. Since $f(0+)$ does not exist, f has the discontinuity of the second kind.

Example 16

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(2\pi x[x])$. If $x \in \mathbb{R} \setminus \mathbb{Z}$, then $x \in (n, n+1)$ for some $n \in \mathbb{Z}$ and $[x] = n$. Thus, $f(x) = \sin(2\pi n x)$ is continuous on x. If $x = n \in \mathbb{Z}$, then

$$\begin{aligned} \lim_{x \to n^+} \sin(2\pi x[x]) = \sin(2\pi n^2) &= 0 \\ \lim_{x \to n^-} \sin(2\pi x[x]) = \sin(2\pi n(n-1)) &= 0 \\ f(n) = \sin(2\pi n^2) &= 0. \end{aligned}$$

Thus, f is continuous on $\mathbb{R}$.

Exercise 10

In the previous example, check that f is not uniformly continuous on $\mathbb{R}$.

Definition 4.10: Monotonically Increasing/Decreasing

A function f is **monotonically increasing/decreasing** if $f(x) \leq f(y)$ or $f(x) \geq f(y)$ $\forall x, y \in I$ with $x < y$.

We call f **monotone** if it is monotonically increasing or decreasing on I.

Definition 4.11: Strictly Increasing/Decreasing

A function f is **strictly increasing/decreasing** if $f(x) < f(y)$ or $f(x) > f(y)$ $\forall x, y \in I$ with $x < y$.

We call f **strictly monotone** if it is strictly increasing or decreasing on I.

Theorem 4.10

Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be a monotone increasing on I. Then $f(p+)$ and $(p-)$ exist for all $p \in I$ and

$$\sup_{x<p} f(x) = f(p-) \leq f(p) \leq f(p+) = \inf_{p<x} f(x).$$

Furthermore, if $p < q$ with $p, q \in I$, then $f(p+) \leq f(q-)$.

Proof. Fix $p \in I$. Since f is increasing, the set $\{f(x) : x < p, x \in I\}$ is bounded above by $f(p)$. Thus, $A = \sum_{x<p} f(x)$ exists in $\mathbb{R}$ and $A \leq f(p)$.

Claim. $A = f(p-)$.

Let $\epsilon > 0$. Since $A = \sup_{x<p}(x)$, $\exists x_0 < p$ such that $A - \epsilon < f(x_0) \leq A$. Since f is increasing, $A - \epsilon < f(x_0) \leq f(x) \leq A$ for all $x \in (x_0, p)$, i.e. $f(p-) = A$. Similarly, $f(p) \leq f(p+) \leq \int_{p<x} f(x)$.

Finally, let $p < q$. then

$$\begin{aligned} f(p+) &= \inf\{f(x) : p < x, x \in I\} \\ &\leq \inf\{f(x) : p < x < q\} \\ &\leq \sup\{f(x) : p < x < q\} \\ &\leq \sup\{f(x) : x < q, x \in I\} \\ &= f(q-). \end{aligned}$$

■

Corollary

If f is monotone on an open interval I, then the set of all discontinuities of f is at most countable.

Proof. Let $E = \{p \in I : f \text{ is discontinuous at } p\}$. We may assume that f is monotone increasing on I. Then $p \in E$ if and only if $f(p-) < f(p+)$. For each $p \in E$, $\exists r_p \in \mathbb{Q}$ such that $f(p-) < r_p < r(p+)$. We define $g : E \to \mathbb{Q}$ by $g(p) = r_p$. If $p < q$ with $p, q \in E$, then $r_p < f(p+) \leq f(q-) < r_q$, and thus g is injective. Therefore, E is equivalent to a subset of $\mathbb{Q}$ and hence it is at most countable. ■

Then, is there a monotone function with at most countable discontinuities? The answer is yes.

Definition 4.12: Unit Jump Function

The function $I(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$ is called the **unit jump function**.

Theorem 4.11

Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable subset of (a, b) and $\{c_n\}_{n\in\mathbb{N}}$ be a sequence such that $c_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n$ converges. Then there exists a monotone increasing function f on $[a, b]$ such that

1. $f(a) = 0$ and $f(b) = \sum_{n=1}^{\infty} c_n$
2. f is continuous on $[a, b] \setminus \{x_n : n \in \mathbb{N}\}$
3. f is right continuous at all x_n, i.e. $f(x_n+) = f(x_n)$ for all $n \in \mathbb{N}$
4. f is discontinuous at each x_n with $f(x_n) - f(x_n-) = c_n$.

Proof. (1) We define $f : [a, b] \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$ First, this funciton is well-defined because $0 \leq c_n I(x - x_n) \leq x_n$, we have

$$s_n(x) = \sum_{k=1}^{n} c_k I(x - x_k) \leq \sum_{k=1}^{n} c_k \leq \sum_{k=1}^{\infty} c_k.$$

Since $\{s_n(x)\}_{n\in\mathbb{N}}$ is increasing and bounded above, $f(x)$ is finite.

For monotonicity, since $I(x - x_n) \leq I(y - x_n)$ for all x and y with $x < y$, f is monotonically increasing on $[a, b]$. Furthermore, since $x_n > a$ for all n, $I(a-x_n) = 0$ for all n. Therefore $f(a) = 0$. Also, since $I(b - x_n) = 1$ for all n, $f(b) = \sum_{k=1}^{\infty} c_k$.

(2) Fix $p \in [a, b]$, $p \neq x_n$ for any n. Let $E = \{x_n : n \in \mathbb{N}\}$.

Suppose p is not a limit point of E. In this case, there exists a $\delta > 0$ such that $N_\delta(p) \cap E = \emptyset$. Then $I(x - x_k) = I(p - x_k)$ for all $x \in (p - \delta, p + \delta)$ and all $k = 1$, $2, \ldots$. Thus f is constant on $(p - \delta, p + \delta)$ and hence continuous.

Suppose p is a limit point of E. Fix $\epsilon > 0$. Since the series $\sum_{k=1}^{\infty} c_k$ converges, $\exists N$

such that $\sum_{k=N+1}^{\infty} c_k < \epsilon$ by Cauchy criterion. Choose δ such that

$$0 < \delta < \min\{|p - x_n| : n = 1, 2, \ldots, N\}.$$

If $x_k \in N_\delta(p) \cap E$, we have $k > N$. Suppose $p < x < p + \delta$. Then $I(p - x_k) = I(x - x_k)$ for all $k = 1, 2, \ldots, N$. Furthermore, for any $x > p$, we always have

$$0 \le I(x - x_k) - I(p - x_k) \le 1, \quad \text{for all} \quad k \in \mathbb{N}.$$

Therefore, for $p < x < p + \delta$,

$$0 \le f(x) - f(p) \le \sum_{k=N+1}^{\infty} c_k \big(I(x - x_k) - I(p - x_k)\big) \le \sum_{k=N+1}^{\infty} c_k < \epsilon.$$

Thus f is right continuous at p. Similarly, f is left continuous a p, and therefore f is continuous at p.

(c) Fix an $x_n \in E$. If x_n is an isolated point of E, then as above, $\exists \delta > 0$ such that $E \cap (x_n, x_n + \delta) = \emptyset$. Therefore $f(y) = f(x_n)$ for all y such that $x_n < y < x_n + \delta$. Thus $f(x_n+) = f(x_n)$.

Suppose x_n is a limit point of E. Fix $\epsilon > 0$. Choose a positive integer N such that $\sum_{k=N+1}^{\infty} c_k < \epsilon$. As above, $\exists \delta > 0$ such that if $x_k \in (x_n, x_n + \delta) \cap E$, then $k > N$. Thus

$$0 \le f(y) - f(x_n) \le \sum_{k=N+1}^{\infty} c_k < \epsilon \quad \text{for all} \quad y \in (x_n, x_n + \delta).$$

Therefore $f(x_n+) = f(x_n)$ and f is right continuous at each x_n.

(d) Suppose $y < x_n$. If x_n is an isolated point of E, $\exists \delta > 0$ such that $(x_n - \delta, x_n) \cap E = \emptyset$. Therefore, for all $k \neq n$, $I(y - x_k) = I(x_n - x_k)$ for all y such that $x_n - \delta < y < x_n$, and for all $y < x_n$,

$$0 = I(y - x_n) \le I(x_n - x_n) = I(0) = 1.$$

Therefore, $f(x_n) - f(y) = c_n$ for all y such that $x_n - \delta < y < x_n$.

Now suppose x_n is a limit point of E. Fix $\epsilon > 0$, and choose N such that $\sum_{k=N+1}^{\infty} c_k < \epsilon$. For this N, choose $\delta > 0$ such that if $x_k \in (x_n - \delta, x_n) \cap E$ then $k > N$. Then for all $y \in [a, b]$ with $x_n - \delta < y < x_n$,

$$c_n \le f(x_n) - f(y) \le c_n + \sum_{k=N+1}^{\infty} c_k < c_n + \epsilon.$$

Therefore, $f(x_n) - f(x_n-) = c_n$. ■

5 Differentiation

5.1 The Derivative

Definition 5.1: Derivative

Let $f : I \to \mathbb{R}$ be a function. For fixed $p \in I$, the **derivative** of f at p, denoted $f'(p)$, is defined by

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p},$$

provided the limit exists. If $f'(p)$ is defined at $p \in I$, we say that f is **differentiable** ot p. If the derivative is defined at every point of a set $E \subset I$, we say that f is differentiable on E.

If p is an interior point of I, then $p + h \in I$ for all sufficiently small h. If we set $x = p + h$, then

$$f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h},$$

provided that the limit exists.

The limit of the formula may only be defined at the left of p, or the right of p. In these cases, we need to define the right and left derivatives.

Definition 5.2: Right and Left Derivative

Let $f : I \to \mathbb{R}$ be a function. If $p \in I$ is such that $I \cap (p, \infty) \neq \emptyset$, then the **right derivative** of f at p, denoted $f'_+(p)$, is defined as

$$f'_+(p) = \lim_{h \to 0^+} \frac{f(p+h) - f(p)}{h},$$

provided the limit exists. Similarly, if $p \in I$ satisfies $(-\infty, p) \cap I \neq \emptyset$, then the **left derivative** of f at p, denoted $f'_-(p)$, is defined as

$$f'_-(p) = \lim_{h \to 0^-} \frac{f(p+h) - f(p)}{h},$$

provided the limit exists.

Remark.

If p is an interior point of I, then $f'(p)$ exists if and only if both $f'_+(p)$ and $f'_-(p)$ exist, and are equal. If $p \in I$ is the left (right) endpoint of I, then $f'(p)$ exists if and only if $f'_+(p)$ $(f'_-(p))$ exists. In this case, $f'(p) = f'_+(p)$ $(f'_-(p))$.

We also use the Leibniz's notations, $\frac{d}{dx}f(x)$, $\frac{df}{dx}$, or $\frac{dy}{dx}$ where $y = f(x)$.

Example 1

Let

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

Then

$$f'_+(0) = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{-h}{h} = -1.$$

Thus $f'_+(0)$ and $f'_-(0)$ both exist, but not equal. Therefore $f'(0)$ does not exist.

Theorem 5.1

If $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is differentiable at $p \in I$, then f is continuous at p.

Proof. For $t \neq p$, $\lim_{t \to p} \frac{f(t) - f(p)}{t - p}$ exists and equals $f'(p)$. Then

$$\lim_{t \to p}(f(t) - f(p)) = \lim_{t \to p}\left(\frac{f(t) - f(p)}{t - p}\right)\lim_{t \to p}(t - p) = f'(p) \cdot 0 = 0.$$

Thus $\lim_{t \to p} f(t) = f(p)$ and thus f is continuous at p. ∎

Exercise 11

Finish the proof when p is an endpoint of I.

Theorem 5.2

Suppose f and g are real-valued functions defined on an interval I. If f and g are differentiable at $x \in I$, then $f + g$, fg, and f/g (if $g(x) \neq 0$) are differentiable at x and

1. $(f + g)'(x) = f'(x) + g'(x)$
2. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
3. $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$, prodivded $g(x) \neq 0$.

Proof. (1) Exercise.

(2) For $h \neq 0$,

$$\frac{(fg)(x+h)-(fg)(x)}{h} = f(x+h)\left(\frac{g(x+h)-g(x)}{h}\right) + \left(\frac{f(x+h)-f(x)}{h}\right)g(x).$$

Since f is differentiable at x, $\lim_{h\to 0} f(x+h) = f(x)$. Thus since each of the limits exist,

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h} \\
&= \lim_{h\to 0} f(x+h) \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right) \\
&+ g(x)\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) \\
&= f(x)g'(x) + g(x)f'(x).
\end{aligned}$$

(3) It is sufficient to prove that $(1/g)'(x) = -g'(x)/(g(x))^2$, provided $g(x) \neq 0$. The result will then follow with (2). If $g(x) \neq 0$, then since g is continuous at x, $g(x+h) \neq 0$ for all sufficiently small h. Thus for sufficiently small and nonzero h,

$$\frac{(1/g(x+h)) - (1/g(x))}{h} = -\left(\frac{g(x+h)-g(x)}{h}\right)\frac{1}{g(x)g(x+h)}.$$

Then

$$\begin{aligned}
\left(\frac{1}{g}\right)'(x) &= \lim_{h\to 0}\frac{(1/g(x+h)) - (1/g(x))}{h} \\
&= \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)\lim_{h\to 0}\frac{1}{g(x)g(x+h)} \\
&= \frac{-g(x)}{(g(x))^2}.
\end{aligned}$$

■

Theorem 5.3: Chain Rule

Suppose f is a real-valued function defined on an interval I and g is a real-valued function defined on some interval J such that the range of f is contained in J. If f is differentiable at $x \in I$ and g is differentiable at $f(x)$, then $h = g \circ f$ is differentiable at x and

$$h'(x) = g'(f(x))f'(x).$$

Proof. For $t \in I$, $t \neq x$, set $Q(x) = (f(t) - f(x))/(t-x)$. Then by the definition,

$Q(t) \to f'(x)$ as $t \to x$. If we let $u(t) = Q(t) - f(x)$, then $u(t) \to 0$ as $t \to x$. Therefore, if f is differentiable at x, for $t \neq x$,

$$f(t) - f(x) = (t - x)(f'(x) + u(t)), \quad \text{where} \quad u(t) \to 0 \text{ as } t \to x.$$

Let $y = f(x)$. Define v similarly with g, so that

$$g(s) - g(y) = (s - y)(g'(y) + v(s)), \quad \text{where} \quad v(s) \to 0 \text{ as } s \to y.$$

Let $s = f(t)$. Since f is continuous at x, $s \to y$ as $t \to x$. We have

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= (f(t) - f(x))(g'(y) + v(s)) \\ &= (t - x)(f'(x) + u(t))(g'(y) + v(f(t))). \end{aligned}$$

Therefore, for $t \neq x$,

$$\frac{h(t) - h(x)}{t - x} = (f'(x) + u(t))(g'(y) + v(f(t))).$$

Since $v(f(t))$ and $u(t)$ both have limit 0 as $t \to x$,

$$\lim_{t \to x} \frac{h(t) - h(x)}{t - x} = f'(x)g'(y) = g'(f(x))f'(x).$$

■

5.2 The Mean Value Theorem

Theorem 5.4: Rolle's Theorem

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Then, $\exists c \in (a, b)$ such that $f'(c) = 0$.

Proof. Since f is continuous on the compact set $[a, b]$, f has a maxumum and a minimum at $[a, b]$.

- If f is constant on $[a, b]$, then $f'(x) = 0$ for $\forall x \in [a, b]$.
- If $f(t) > f(a)$ for some t, then f has a maximum at $c \in (a, b)$. At this point, we have $f'(c) = 0$.
- If $f(t) < f(a)$ for some t, then f has a minimum at $c \in (a, b)$, and $f'(c) = 0$.

■

Remark.

The continuity of f on $[a, b]$ is requied.

Remark.
Differentiability of f at a and b is not required. Consider $f(x) = \sqrt{4-x^2}$, where $x \in [-2, 2]$. It is not differentiable at $x = \pm 2$, but $f'(0) = 0$.

Theorem 5.5: Mean Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

Proof. Consider the function

$$g(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right).$$

Then g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = g(b) = 0$. By Rolle's theorem, $\exists c \in (a, b)$ such that $g'(c) = 0$, i.e.

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

■

Example 2

Show that $\frac{x}{1+x} \le \ln(1 + x) \le x$ for $x > -1$.

Solution Let $f(x) = \ln(1 + x)$ where $x \in (-1, \infty)$. If $x > 0$, the MVT shows that $\exists c \in (0, x)$ such that

$$\ln(1 + x) = f(x) - f(0) = f'(c)x.$$

But $f'(c) = \frac{1}{1+c}$ and $\frac{1}{1+x} < \frac{1}{1+c} < 1$. Therefore,

$$\frac{x}{1+x} \le \ln(1 + x) \le x \ \forall x \ge 0.$$

If $x \in (-1, 0)$, then by MVT again, $\exists c \in (x, 0)$ such that $\ln(1 + x) = \frac{x}{1+c}$. In this case, $1 < \frac{1}{1+c} < \frac{1}{1+x}$ and hence $\frac{x}{1+x} < \ln(1 + x) < x$ for $\forall x \in (-1, 0)$.

Theorem 5.6: Cauchy's Mean Value Theorem

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists c \in (a, b)$ such that

$$\big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c).$$

Proof. Let $h(x) = \big(f(b) - f(a)\big)g(x) - \big(g(b) - g(a)\big)f(x)$. Since h is continuous on $[a, b]$ and differentiable on (a, b), it follows from Rolle's theorem that $\exists c \in (a, b)$ such that $h'(c) = 0$. ■

Applications

Theorem 5.7

Let $f : I \to \mathbb{R}$ be differentiable on an interval I.

- If $f'(x) \geq 0$ for $\forall x \in I$, then f is monotonically increasing on I.
- If $f'(x) \leq 0$ for $\forall x \in I$, then f is monotonically decreasing on I.
- If $f'(x) > 0$ for $\forall x \in I$, then f is strictly increasing on I.
- If $f'(x) < 0$ for $\forall x \in I$, then f is strictly decreasing on I.

Proof. Suppose x_1 and $x_2 \in I$ with $x_1 < x_2$. By the MVT applied to $[x_1, x_2]$, $\exists c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1)f'(c)(x_2 - x_1).$$

If $f'(c) \geq 0$, then $f(x_2) \geq f(x_1)$. Thus, f is monotone increasing on I. The other results follow similarly. ■

Remark.

If $f'(c) > 0$ at one point $c \in I$, then $\exists \delta > 0$ such that $f(x) < f(c)$ for $\forall x \in (c - \delta, c)$ and $f(c) < f(x)$ for $\forall x \in (c, c + \delta)$. However, this does not imply that f is increasing on $(c - \delta, c + \delta)$.

Consider the function

$$f(x) = \begin{cases} x + x^2 \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

This function satisfies $f'(0) = 1 > 0$, but f' has both positive and negative values in every neighborhood of 0.

Corollary

If $f'(x) = 0$ for $\forall x \in I$, then f is constant on I.

Theorem 5.8

If $f'(c) > 0$ and f' is continuous at c, then $\exists\delta > 0$ such that $f'(x) > 0$ for $\forall x \in (c-\delta, c+\delta)$. In this case, f is strictly increasing on $(c-\delta, c+\delta)$.

If f has a local minimum at c, then is f decreasing to the left of c and increasing to the right of c? The answer is no. Consider the function

$$f(x) = \begin{cases} x^4\left(2+\sin\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Then f has an absolute minimum at 0. But, f' has both positive and negative values in every neighborhood of 0.

Theorem 5.9

Suppose that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If $\lim\limits_{x\to a^+} f'(x)$ exists, then $f'_+(a)$ exists and

$$f'_+(a) = \lim_{x\to a^+} f'(x).$$

Proof. Let $L = \lim\limits_{x\to a^+} f'(x)$. Given $\epsilon > 0$, $\exists\delta \in (0, b-a)$ such that

$$|f'(x) - L| < \epsilon \quad \text{for all} \quad x \in (a, a+\delta).$$

Let $h \in (0,\delta)$, then f is continuous on $[a, a+h]$ and differentiable on $(a, a+h)$. By the MVT,

$$f(a+h) - f(a) = f'(c_h)h$$

for some $c_h \in (a, a+h)$. Therefore,

$$\left|\frac{f(a+h)-f(a)}{h} - L\right| = |f'(c_h) - L| < \epsilon.$$

■

Theorem 5.10: Intermediate Value Theorem of Derivatives

Suppose that $f : I \to \mathbb{R}$ is differentiable on I. Let $a, b \in I$ be such that $a < b$. If $f'(a) \neq f'(b)$ and $\lambda \in \mathbb{R}$ is in between $f'(a)$ and $f'(b)$, then $\exists c \in (a,b)$ such that $f'(c) = \lambda$.

Note The theorem above does not require continuity of f'.

Proof. Define $g(x) = f(x) - \lambda x$. Since g is continuous on $[a,b]$, g attains an absolute minimum at some point $c \in [a,b]$. Suppose that $f'(a) < \lambda < f'(b)$. Then $g'(a) < 0$ and $g''(b) > 0$. Thus, $\exists x_1 > a$ such that $g(x_1) < g(a)$ and $\exists x_2 < b$

such that $g(x_2) < g(b)$. This implies that $c \neq a$ and $c \neq b$, and hence $c \in (a, b)$. Moreover, since g is differentiable on $[a, b]$, we have $g'(c) = 0$, i.e. $f'(c) = \lambda$. ■

Theorem 5.11: Inverse Function Theorem

Let $f : I \to \mathbb{R}$ be differentiable on I with $f'(x) \neq 0$ for $\forall x \in I$. Then $f : I \to f[I]$ is invertible. Moreover, the inverse function $f^{-1} : f[I] \to I$ is differentiable (and hence continuous) on $f[I]$ and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)} \text{ for all } x \in I.$$

Proof. Since $f' \neq 0$ on I, the IVT for derivatives shows that f' is either positive on I or negative on I. Assume $f' > 0$ on I. Then f is strictly increasing on I, and therefore f^{-1} exists and continuous on $f[I]$.

To show that f^{-1} is differentiable on $f[I]$, let $y_0 \in f[I]$ and let $\{y_n\} \subseteq f[I]$ be any sequence with $y_n \to y_0$ as $n \to \infty$, and $y_n \neq y_0$ for all $n \in \mathbb{N}$. Then, $\exists x_n \in I$ such that $f(x_n) = y_n$. Since f^{-1} is continuous, $x_n \to x_0 = f^{-1}(y_0)$ as $n \to \infty$. We have

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{f'(x_0)}.$$

Since y_0 and $\{y_n\}$ were arbitrary, we conclude the theorem. ■

5.3 L'Hospital's Rule

Definition 5.3: Diverge to ∞

Let $E \subseteq \mathbb{R}$, $f : E \to \mathbb{R}$, and $p \in E'$. We say that f **diverges to** ∞ as $x \to p$, if $\forall M \in \mathbb{R}$, $\exists \delta > 0$ such that $f(x) > M$ for all $x \in E$ with $0 < |x - p| < \delta$. We use the notation

$$\lim_{x\to p} f(x) = \infty.$$

We use a similar definition for $-\infty$.

Theorem 5.12: L'Hospital's Rule

Let $-\infty \leq a < b \leq \infty$ and $L \in [-\infty, \infty]$. Let $f,\ g : (a,b) \to \mathbb{R}$ be differentiable on (a,b) and $g' \neq 0$ on (a,b). Suppose that

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L.$$

If

1. $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^+} g(x) = 0$ or
2. $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^+} g(x) = \pm\infty$.

Then $\lim\limits_{x \to a^+} \dfrac{f(x)}{g(x)} = L$.

The analogous results where $x \to b^-$ is also true.

Proof. Supposde (1) holds. We first prove the case where a is finite. Let $\{x_n\}$ be a sequence in (a,b) with $x_n \to a$ and $x_n \neq a$ for all n. Setting $f(a) = g(a) = 0$ gives f and g continuous at a. Thus for each $n \in \mathbb{N}$, there exists c_n between a and x_n such that

$$(f(x_n) - f(a))g'(c_n) = (g(x_n) - g(a))f'(c_n),$$

or

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(c_n)}{g'(c_n)}.$$

Since $g'(x) \neq 0$ for all $x \in (a,b)$, $g(x_n) \neq g(a)$ for all n. As $n \to \infty$, $c_n \to a^+$. Thus

$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = \lim_{n \to \infty} \frac{f'(c_n)}{g'(c_n)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L.$$

Since the above holds for every sequence $\{x_n\}$ with $x_n \to a^+$, the result follows.

Suppose $a = -\infty$, and let $x = -1/t$. Then as $t \to 0^+$, $x \to -\infty$. Define the functions $\varphi(t)$ and $\psi(t)$ on $(0,c)$ for some $c > 0$ by $\varphi(t) = f(-1/t)$ and $\psi(t) = g(-1/t)$. Then

$$\lim_{t \to 0^+} \frac{\varphi'(t)}{\psi'(t)} = \lim_{x \to -\infty} \frac{f'(x)}{g'(x)} = L$$

with $\lim\limits_{t \to 0^+} \varphi(t) = \lim\limits_{t \to 0^+} \psi(t) = 0$. Thus

$$\lim_{x \to -\infty} \frac{f(x)}{g(x)} = \lim_{t \to 0^+} \frac{\varphi(t)}{\psi(t)} = L.$$

Now suppose (2) holds, i.e. $\lim\limits_{x \to a^+} g(x) = \infty$. The case where $g(x) \to -\infty$ is treated similarly. Suppose first that $-\infty \leq L < \infty$, and $\beta \in \mathbb{R}$ satisfies $\beta > L$. Choose r

such that $L < r < \beta$. Since
$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} < r,$$
there exists $c_1 \in (a, b)$ such that $f'(\zeta)/g'(\zeta) < r$ for all $\zeta \in (a, c_1)$.

Fix $y \in (a, c_1)$. Since $g(x) \to \infty$ as $x \to a^+$, there exists $c_2 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ for all $x \in (a, c_2)$. Let $x \in (a, c_2)$ be arbitrary. Then by the generalized MVT, there exists $\zeta \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\zeta)}{g'(\zeta)} < r.$$

Since $g(x) > g(y)$ and $g(x) > 0$, we have $(g(x) - g(y))/g(x) > 0$. Multiplying this gives

$$\begin{aligned}\frac{f(x) - f(y)}{g(x)} &< r\left(1 - \frac{g(y)}{g(x)}\right)\\ \frac{f(x)}{g(x)} &< \frac{f(y)}{g(x)} + r\left(1 - \frac{g(y)}{g(x)}\right)\end{aligned}$$

for all $x \in (a, c_2)$. Now for fixed y, since $g(x) \to \infty$,

$$\lim_{x\to a+} \frac{f(y)}{g(x)} = \lim_{x\to a^+} \frac{g(y)}{g(x)} = 0.$$

Therefore
$$\lim_{x\to a^+} \frac{f(y)}{g(x)} + r\left(1 - \frac{g(y)}{g(x)}\right) = r < \beta.$$
Thus there exists $c_3 \in (a, c_2)$ such that

$$\frac{f(y)}{g(x)} + r\left(1 - \frac{g(y)}{g(x)}\right) < \beta$$

for all $x \in (a, c_3)$. Thus $f(x)/g(x) < \beta$ for all $x \in (a, x_3)$.

If $L = -\infty$, then for any $\beta \in \mathbb{R}$, there exists c_3 such that the formula above holds for all $x \in (a, c_3)$. Thus by definition, $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)} = -\infty$.

If L is finite, then given $\epsilon > 0$, by taking $\beta = L + \epsilon$, there exists c_3 such that $f(x)/g(x) < L + \epsilon$ for all $x \in (a, c_3)$.

Suppose $-\infty < L \leq \infty$. Let $\alpha \in \mathbb{R}$, $\alpha < L$ be arbitrary. Then an argument similar to the above gives the existence of $c_3' \in (a, b)$ such that $f(x)/g(x) > \alpha$ for all $x \in (a, c_3')$.

If $L = \infty$, then this implies that $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)} = \infty$.

If L is finite, taking $\alpha = L-\epsilon$ gives the existence of a c_3' such that $f(x)/g(x) > L-\epsilon$

for all $x \in (a, c_3')$. Combining, we have

$$L - \epsilon < \frac{f(x)}{g(x)} < L + \epsilon,$$

and therefore

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L. \qquad \blacksquare$$

Example 3

$$\lim_{x \to 0^+} \frac{\ln(1+x)}{x} = \lim_{x \to 0^+} \frac{\frac{1}{1+x}}{1} = 1.$$

Example 4

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Example 5

If we apply L'Hospital's rule to $\lim_{x \to 0^+} \frac{e^{-1/x}}{x}$, then we get $\lim_{x \to 0^+} \frac{e^{-1/x}}{x^2}$. Instead, if we take $t = 1/x$, then we get

$$\lim_{x \to 0^+} \frac{e^{-1/x}}{x} = \lim_{t \to \infty} \frac{t}{e^t} = \lim_{t \to \infty} \frac{1}{e^t} = 0.$$

5.4 Newton's Method

Newton-Rhapson method is a root-finding algorithm producing approximations to the roots of $f(x) = 0$.

We first state the bisection method. Take some a and b, and if $f(a)f(b) < 0$, take $c = (a+b)/2$.

- If $f(c) = 0$, then we are done.
- If $f(c) \neq 0$, then either $f(a)f(c) < 0$, and $f(c)f(b) < 0$.

We repeat the process. This algorithm is used to approximate the root, but it's slow.

For the Newton-Rhapson method, assume that $f(a)f(b) < 0$ and $f' \neq 0$ on $[a, b]$. Let c_1 be an initial guess. Then the tangent line

$$y = f(c_1) - f'(c_1)(x - c_1)$$

crosses the x-axis at $c_2 = c_1 - \frac{f(c_1)}{f'(c_1)}$. We repeat the process.

Lemma

Let $f : [a, b] \to \mathbb{R}$ be such that f' is continuous on $[a, b]$ and f'' exists on (a, b). Let $x_0 \in [a, b]$. Then for any $x \in [a, b]$, $\exists \zeta$ between x_0 and x such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(\zeta)(x - x_0)^2.$$

Proof. Fix $x \in [a, b]$. If $x = x_0$, then any ζ works. Otherwise, $\exists \alpha \in \mathbb{R}$ such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \alpha(x - x_0)^2.$$

Define $g : [a, b] \to \mathbb{R}$ by

$$g(t) = f(t) - \big(f(x_0) + f'(x_0)(t - x_0) + \alpha(t - x_0)^2\big).$$

We may assume $x > x_0$. Since g is continuous and differentiable on $[x_0, x]$ and $g(x_0) = g(x) = 0$, it follows from Rolle's theorem that $\exists c_1 \in (x_0, x)$ such that $g'(c) = 0$. But

$$g'(t) = f'(t) - f'(x_0) - 2\alpha(t - x_0).$$

Since g is continuous on $[x_0, c]$, differentiable on (x_0, c), and $g'(x_0) = 0 = g'(c)$, by Rolle's theorem again, $\exists \zeta \in (x_0, c)$ such that $g''(\zeta) = 0$. Therefore, $f''(\zeta) - 2\alpha = 0$, and $\alpha = \frac{1}{2}f''(\zeta)$. ■

Theorem 5.13: Newton's Method

Let $f : [a, b] \to \mathbb{R}$ be twice differentiable on $[a, b]$. Suppose that $f(a)f(b) < 0$ and that $\exists m, M > 0$ such that $|f'| \geq m$ and $|f''| \leq M$. Then there exists a subinterval $I \subseteq [a, b]$ and a zero $c \in I$ such that $f(c) = 0$.

For all $c_1 \in I$, the sequence $\{c_n\}$ defined by

$$c_{n+1} = c_n - \frac{f(c_n)}{f'(c_n)} \quad n \in \mathbb{N}$$

is in I, and $\lim_{n\to\infty} c_n = c$. Furthermore,

$$|c_{n+1} - c| \leq \frac{M}{2m}|c_n - c|^2.$$

Proof. Since $f(a)f(b) < 0$ and $f' \neq 0$ on $[a, b]$, f has exactly one zero $c \in (a, b)$. Let $x_0 \in [a, b]$. By the lemma, $\exists \zeta$ between c and x_0 such that

$$0 = f(c) = f(x_0) + f'(x_0)(c - x_0) + \frac{1}{2}f''(\zeta)(c - x_0)^2,$$

or

$$-f(x_0) = f'(x_0)(c - x_0) + \frac{1}{2}f''(\zeta)(c - x_0)^2.$$

Define $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$. Then

$$x_1 = x_0 + (c - x_0) + \frac{1}{2}\frac{f''(\zeta)}{f'(x_0)}(c - x_0)^2$$

and hence

$$|x_1 - c| = \frac{1}{2}\frac{|f''(\zeta)}{|f'(x_0)|} \leq \frac{M}{2m}|c - x_0|^2.$$

Choose $\delta > 0$ sufficiently small so that $\delta < \dfrac{2m}{M}$ and $I = [c - \delta, c + \delta] \subseteq [a, b]$. If $c_n \in I$, then $|c - c_n| < \delta$. Thus, if $c_{n+1} = c_n - \dfrac{f(c_n)}{f'(c_n)}$, then

$$|c_{n+1} = c| \leq \frac{M}{2m}\delta^2 < \delta,$$

i.e. $c_{n+1} \in I$. This means that if $c_1 \in I$, then $c_n \in I$ for all $n \in \mathbb{N}$. Moreover, since

$$|c_{n+1} - c| < \frac{M}{2m}\delta|c_n - c| < \cdots < \left(\frac{M}{2m}\delta\right)^n |c_1 - c|$$

and $\dfrac{M}{2m}\delta < 1$, we conclude that $c_n \to c$. ■